Solution of the Potential Difference Integral
by Using Exponential Windowing

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INTRODUCTION

The purpose of this report is to present a brief description of the methodology used for solving the potential difference integral in the electric logging problem. The description is mainly focused on the mathematical manipulations that are performed in order to make the numerical solution of the integral more suitable from a computational point of view. Details of the computational algorithms are not provided at this moment.

SEPARATION IN TWO INTEGRALS

The integral we are interested in solving, given in equation (52) of [1], can be written as follows:

$\Delta R(z)$ = **¡Error! ¡Error!** (1)

where $ß = ß(\lambda) = \sqrt{2, \lambda^2 + j\omega\sigma}$ (2)

r_0 is the radius of the logging tool, h is the segment length, ω is the angular frequency, σ is the electric conductivity of the zone 1, Γ is the reflection factor in zone 1 and K_0 and I_0 are the zero order modified Bessel functions of second and first kind.

By using the following two complementary exponential windows:

$w_1(\lambda)$ = Exp **¡Error!** (3.a)

$w_2(\lambda)$ = 1 - Exp **¡Error!** (3.b)

The integral in (1) can be separated in two integrals as follows:

$\Delta R(z) =$ **¡Error!** $[I_1(z) + I_2(z)]$ (4)

where:

$I_1(z) =$ **¡Error! ¡Error!** (5.a)

$I_2(z) =$ **¡Error! ¡Error!** (5.b)

COSINE TRANSFORM INTEGRAL

It is possible to define the two functions $G_1(\lambda)$ and $F_1(\lambda)$ as:

$G_1(\lambda) =$ ß **¡Error!** Exp**¡Error!** (6)

$F_1(\lambda) =$ **¡Error!** (7)

so that the integral in (5.a) can be rewritten as:

$I_1(z) =$ **¡Error! ¡Error!** (8)

It can be seen from (8) that $I_1(z)$ is the inverse fourier transform of the product of the two functions $G_1(\lambda)$ and $F_1(\lambda)$. By using the product property of the fourier transform, $I_1(z)$ can be calculated by a linear convolution as follows:

$$I_1(z) = g_1(z) * f_1(z) \quad (9)$$

where $g_1(z)$ and $f_1(z)$ are the inverse fourier transforms of $G_1(\lambda)$ and $F_1(\lambda)$ respectively.

The function $f_1(z)$ can be computed analytically and is given by:

$$f_1(z) = \begin{cases} 1/16\, [\, z^2 + 3hz + 9h^2/4\,] & \text{for } -3h/2 \le z \le -h/2 \\ 1/16\, [\, -2z^2 + 3h^2/2\,] & \text{for } -h/2 \le z \le +h/2 \\ 1/16\, [\, z^2 - 3hz + 9h^2/4\,] & \text{for } +h/2 \le z \le +3h/2 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

while the function $g_1(z)$ has to be computed numerically. Due to the even symmetry of $G_1(\lambda)$, $g_1(z)$ can be written as:

$g_1(z) =$ **¡Error! ¡Error!** (11)

Notice that because the oscillatory behavior of the integrand due to $F_1(\lambda)$ has been removed and that because the slope has been improved by the exponential window, the integral in (11) can be nicely solved by using the Anderson's integration technique [2].

SINE TRANSFORM INTEGRAL

Proceeding in the same way as in the cosine transform integral, two functions $G_2(\lambda)$ and $F_2(\lambda)$ can be defined as follows:

$G_2(\lambda)$ = **¡Error! ¡Error! ¡Error!** (12)

$F_2(\lambda)$ = **¡Error!** (13)

so that the integral in (5.b) can be rewritten as:

$I_2(z)$ = **¡Error! ¡Error!** (14)

and as in the previous case, $I_2(z)$ can be expressed by the linear convolution of the inverse fourier transforms of $G_2(\lambda)$ and $F_2(\lambda)$; they are $g_2(z)$ and $f_2(z)$ respectively.

Similarly, the function $f_2(z)$ can be computed analytically and is given by:

$f_2(z) = -j/8\,[\,z + 3h/2\,]$	for	$-3h/2 \le z \le -h/2$
$j\,z/4$	for	$-h/2 \le z \le +h/2$
$-j/8\,[\,z - 3h/2\,]$	for	$+h/2 \le z \le +3h/2$
0	otherwise	(15)

while the function $g_2(z)$ has to be computed numerically. Due to the odd symmetry of $G_2(\lambda)$, $g_2(z)$ can be written as:

$g_2(z)$ = **¡Error! ¡Error!** (16)

where the Anderson's integration technique [2] can be used again.

However, there is a practical problem with this last integral. It happens to be that $g_2(z)$ tends to infinity when z approaches zero, which is due to the behavior of $G_2(\lambda)$ for large values of λ. As it can be verified $G_2(\lambda) \to -1$ as $\lambda \to \infty$. This fact is responsible for big computational errors when evaluating the discrete convolution of $g_2(z)$ with $f_2(z)$ for values of z that are smaller than 3h/2.

Hopefully, this problem can be solved by removing from the integral the contribution due to $G_2(\lambda)$ when λ goes to infinity. It can be done as follows:

$g_2(z)$ = **¡Error! ¡Error! - ¡Error! ¡Error!** (17)

where the first integral is going to be referred as $g_2^+(z)$ and the second one as $g_2^-(z)$. Now the integral $I_2(z)$ can be expressed in terms of the two new integrals as:

$$I_2(z) = g_2^+(z) * f_2(z) + g_2^-(z) * f_2(z) \quad (18)$$

It can be verified that there is an analytical solution for $g_2^-(z)$. It is given by:

$g_2^-(z)$ = **- ¡Error! ¡Error! = - ¡Error!** (19)

and its convolution with $f_2(z)$ can also be computed analytically and is given by:

$\xi(z) = g_2^-(z) * f_2(z)$ = **¡Error!** ln**¡Error! - ¡Error!** ln**¡Error! - ¡Error!** ln**¡Error!** (20)

Notice that special care must be taken during the numerical evaluation of $\xi(z)$ at z = -3h/2, -h/2, h/2 and 3h/2.

FINAL REVIEW

Finally, the original integral presented in (1) can be computed as:

$\Delta R(z)$ = **¡Error!** $[\, g_1(z) * f_1(z) + g_2^+(z) * f_2(z) + \xi(z) \,]$ (21)

where $f_1(z)$, $f_2(z)$ and $\xi(z)$ are known functions given by equations (10), (15) and (20) respectively, while $g_1(z)$ and $g_2^+(z)$ are given by:

$g_1(z)$ = **¡Error! ¡Error!** (22)

$g_2^+(z)$ = **¡Error! ¡Error!** (23)

and can be evaluated by using the Anderson's integration technique [2].

REFERENCES

[1] Bostick, F.; Smith, H. (1994), Propagation Effects in Electric Logging. University of Texas at Austin.

[2] Anderson, W. (1995), Improved Digital Filters for Evaluating Fourier and Hankel Transform Integrals. U.S.G.S. rep. USGS-GD-75-012.