Asymptotic Approximation for the Potential Difference Integral

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INTRODUCTION

This report presents an attempt to compute an asymptotic approximation for the integral involved in the solution of the time harmonic field electric logging problem presented in [1]. Due to the huge computation time required to evaluate that integral by conventional procedures, and due to the failure of more advanced integration techniques, we are attempting to find such approximation. The availability of an asymptotic approximation not only would simplify the numerical solution of the problem, but also would help for a better understanding of the problem.

FIRST ATTEMPT: THE MELLIN TRANSFORM METHOD

The integral we are interested in computing is presented in [1] and is given by:

$$\Delta R(z) = \frac{4}{r_0\,\sigma\,h\,\pi^2} \int_0^\infty ß \left[\frac{K_0(ß\,r_0) + \Gamma\, I_0(ß r_0)}{K_0'(ß\,r_0) + \Gamma\, I_0'(ß r_0)} \right] \frac{\mathrm{Sin}^3(\lambda h/2)}{\lambda^3}\, \mathrm{Cos}(\lambda z)\, d\lambda \qquad (1)$$

where $ß = ß(\lambda) = \pm\sqrt{\lambda^2 + j\omega\mu\,\sigma}$, r_0 is the radius of the logging tool, h is the segment length, ω is the angular frequency of operation, σ is the conductivity, Γ is the reflection coefficient and I_0 and K_0 are the zero order Modified Bessel functions of first and second kind.

For simplicity, let us first consider the homogeneous problem in which Γ_1 is equal to zero. So, (1) becomes:

$$\Delta R(z) = \frac{4}{r_0\,\sigma\,h\,\pi^2} \int_0^\infty ß \left[\frac{K_0(ß\,r_0)}{K_0'(ß\,r_0)} \right] \frac{\mathrm{Sin}^3(\lambda h/2)}{\lambda^3}\, \mathrm{Cos}(\lambda z)\, d\lambda \qquad (2)$$

What we basically want to do is to find an approximation of $\Delta R(z)$ for large values of z.

The integral in (2) belongs to a general class of integrals of the form:

$$I(z) = \int_a^b f(\lambda)\ h(z\,\phi(\lambda))\,d\lambda \tag{3}$$

where:

$$f(\lambda) = ß\left[\frac{K_0(ß\,r_0)}{K_0'(ß\,r_0)}\right]\frac{Sin^3(\lambda h/2)}{\lambda^3}, \tag{4}$$

$$\phi(\lambda) = \lambda\,, \tag{5}$$

and $h(z\,\phi(\lambda)) = Cos(z\lambda)$ (6)

According to the Mellin Transform Method [2], the possible critical points for an asymptotic expansion of (3) are given by:

* The endpoints of integration; that is $\lambda = 0$ and $\lambda = \infty$.
* Those values of λ where $\phi(\lambda)$ or $f(\lambda)$ are not infinitely differentiable.
* Those values of λ where $\phi(\lambda)$ or its derivative vanish, which happens to be at $\lambda = 0$.

It is clear from (2) that $\lambda = \infty$ is not a critical point. That is because (6) oscillates faster and faster when λ goes to infinity. So, the contributions to the final value of the integral are smaller as λ increases. Then, the only critical point seems to be $\lambda = 0$.

By following the procedure described in [2], a possible expansion can be given by:

$$I(z)\Big|_{z\to\infty} \approx \sum_{m=0}^{\infty} z^{-1-a_m} \sum_{n=0}^{\overline{N}(m)} p_{mn} \sum_{j=0}^{n} \binom{n}{j} (-\log z)^j\ M^{(n-j)}[h;s]\Big|_{s=1+a_m} \tag{7}$$

where the coefficients a_m and p_{mn} are defined in [2] and $M[h;s]$ is the Mellin Transform of h(t), which in our case is given by:

$$M[Cos(t);s] = \Gamma(s)\,Cos(\pi s/2) \tag{8}$$

where $\Gamma(s)$ is the gamma function.

The approximation given by (7) can be simplified even more by assuming:

$$f(\lambda)\Big|_{\lambda\to 0} \approx \gamma_0\,\lambda^{\mu_0} \quad \text{and} \quad \phi(\lambda)\Big|_{\lambda\to 0} \approx \alpha_0\,\lambda^{\nu_0} \quad \alpha_0, \nu_0 > 0 \tag{9}$$

If (9) holds, then (7) becomes:

$$I(z)\big|_{z\to\infty} \approx (\alpha_0 z)^{-(1+\mu_0)/\nu_0} \frac{\gamma_0}{\nu_0} M[h;(1+\mu_0)/\nu_0] \tag{10}$$

By using (8) and $\alpha_0 = \nu_0 = 1$, which follows from (5) and (9), into (10) we get:

$$I(z)\big|_{z\to\infty} \approx z^{-(1+\mu_0)} \gamma_0 \Gamma(1+\mu_0) \mathrm{Cos}(\pi(1+\mu_0)/2) \tag{11}$$

It still remains to verify if the function given in (4) can be expressed as in (9). If that happens to be the case, then (11) would be the approximation we are looking for.

First, let us consider the D.C. case, in which ß = λ and (4) becomes:

$$f(\lambda) = \left[\frac{K_0(\lambda r_0)}{K_0^{'}(\lambda r_0)}\right] \frac{\mathrm{Sin}^3(\lambda h/2)}{\lambda^2} \tag{12}$$

By using the following approximations:

$$K_0(x)\big|_{x\to 0} \approx -\ln x, \tag{13.a}$$

$$K_0^{'}(x)\big|_{x\to 0} \approx -\frac{1}{x}, \tag{13.b}$$

and $\mathrm{Sin}(x)\big|_{x\to 0} \approx x$ (13.c)

we get:

$$f(\lambda)\big|_{\lambda\to 0} \approx \frac{h^3 r_0}{8} \lambda^2 \ln(\lambda r_0) \tag{14}$$

where it is evident that f(λ) cannot be expressed as in (9). Therefore the approximation given by (11) cannot be used.

Now, let us see if (11) could be valid for the high frequency case. For a very high frequency, it can be seen that $ß \approx \gamma = \sqrt{j\omega\mu\sigma}$. Then, f(λ) is given by constant and (11) reduces to zero. Again, the approximation given by (11) cannot be used.

There is still another simplification for (7) when functions with logarithmic behavior are involved. Notice from (14) that in the D.C. case, f(λ) presents a logarithmic behavior for λ -> 0. Again, by following the procedure described in [2], if f(λ) can be approximated by:

$$f(\lambda)\big|_{\lambda\to 0} \approx \gamma_{01}\,\lambda^{\mu_0}\ln\lambda + \gamma_{00}\,\lambda^{\mu_0} \tag{15}$$

then the asymptotic expansion given by (7) reduces to:

$$I(z)\big|_{z\to\infty} \approx z^{-(1+\mu_0)}\left\{\gamma_{01}\ln z\, M[h,1+\mu_0] - \gamma_{00}\,M[h,1+\mu_0] - \gamma_{01}\frac{d}{ds}M[h,s]\bigg|_{s=1+\mu_0}\right\} \tag{16}$$

By comparing (14) with (15) we obtain:

$$\gamma_{01} = \frac{h^3\,r_0}{8};\quad \gamma_{00} = 0 \quad \text{and} \quad \mu_0 = 2 \tag{17}$$

Now, by using (8), we can compute the terms in (16). They are:

$$M[h;1+\mu_0] = M[h;3] = \Gamma(3)\,\mathrm{Cos}(3\pi/2) = 0 \tag{18.a}$$

$$\frac{d}{ds}M[h,s]\bigg|_{s=1+\mu_0} = \frac{d}{ds}\{\Gamma(s)\,\mathrm{Cos}(s\pi/2)\}\bigg|_{s=1+\mu_0} = \left\{\left[\frac{d}{ds}\Gamma(s)\right]\mathrm{Cos}(s\pi/2) - \frac{\pi}{2}\,\mathrm{Sin}(s\pi/2)\,\Gamma(s)\right\}\bigg|_{s=1+\mu_0}$$

$$\Rightarrow \frac{d}{ds}M[h,s]\bigg|_{s=3} = -\frac{\pi}{2}\,\mathrm{Sin}(3\pi/2)\,\Gamma(3) = \pi \tag{18.b}$$

By replacing (17) and (18) into (16):

$$I(z)\big|_{z\to\infty} \approx \frac{-\pi\,r_0\,h^3 z^{-3}}{8} \tag{19}$$

Finally, an asymptotic expansion for ΔR(z) is obtained by substituting (19) into (2):

$$\Delta R(z)\big|_{z\to\infty} \approx \frac{-h^2\,z^{-3}}{2\pi\sigma} \tag{20}$$

Notice that this approximation is only valid for the homogeneous D.C. case.

SECOND ATTEMPT: THE ELECTRIC DIPOLE

After inspecting (20), its great similarity to the far field generated by an electric dipole can be noticed. For this reason, we are going to attempt a new approximation of (2) starting from the field generated by an electric dipole.

The magnetic and electric fields generated in an homogeneous dielectric medium by a infinitesimal current element of length dl and strength I Cos(ωt) are given by:

$$\overline{H}(r,\theta,\phi) = \frac{I\,dl}{4\pi}\,\mathrm{Sin}\theta\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)e^{-j\beta r}\,\overline{a}_\phi \tag{21}$$

$$\overline{E}(r,\theta,\phi) = -j\frac{\eta\,I\,dl}{2\pi\beta}\,\mathrm{Cos}\theta\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{-j\beta r}\,\overline{a}_r - j\frac{\eta\,I\,dl}{4\pi\beta}\,\mathrm{Sin}\theta\left(-\frac{\beta^2}{r}+\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{-j\beta r}\,\overline{a}_\theta \tag{22}$$

where $\beta = \omega\sqrt{\mu\,\varepsilon}$ and $\eta = \sqrt{\mu/\varepsilon}$.

A full description of how these fields are computed is presented in [3].

Now, we are interested in approximate ΔR(z). It can be verified from [1], that the expression for ΔR(z) given in (2) was obtained from the following definition:

$$\Delta R(z) = \frac{1}{I}\int_{z-h/2}^{z+h/2}\overline{E}_z(z,r_0)\,d\overline{z} \tag{23}$$

where $\overline{E}_z(z,r_0)$ is the z component of the electric field generated by the current element along the tool surface, I is its current strength and h is the segment length.

By considering that the expansion we are looking for is for large values of z and that r_0 is relatively small compared to those values of z, we can make the following approximations in (21) and (22):

$$\theta \approx 0 \quad \text{and} \quad r \approx z \tag{24}$$

which yields to:

$$\overline{H}(z,0) = 0 \tag{25}$$

$$\bar{E}(z,0) = -j\frac{\eta I\, dl}{2\pi\beta}\left(\frac{j\beta}{z^2}+\frac{1}{z^3}\right) e^{-j\beta z}\ \bar{a}_z \tag{26}$$

In addition, by considering that we are interested in the field generated in a conductive medium instead of a dielectric medium, let us make the following intuitive substitutions:

$$j\beta = \gamma \tag{27.a}$$

$$\eta = \frac{\gamma}{\sigma} \tag{27.b}$$

where $\gamma = \sqrt{j\omega\mu\sigma}$ and σ is the conductivity of the medium. Then, (26) reduces to:

$$\bar{E}(z,0) = \frac{I\,h}{2\pi\sigma}\left(\frac{\gamma}{z^2}+\frac{1}{z^3}\right) e^{-\gamma z}\ \bar{a}_z \tag{28}$$

Notice also that the infinitesimal current element length dl has been replaced with the segment length h.

Finally, considering that:

$$\bar{E}_z(z, r_0)\Big|_{z\to\infty} \approx \bar{E}(z,0) \tag{29}$$

we can compute the desired approximation by replacing (28) into (23). By doing that we get:

$$\Delta R(z)\Big|_{z\to\infty} \approx \frac{h}{2\pi\sigma}\left[\gamma\, I_2(z) + I_3(z)\right] = \frac{h}{2\pi\sigma}\left[\gamma \int_{z-h/2}^{z+h/2} \frac{e^{-\gamma z}}{z^2}\, dz + \int_{z-h/2}^{z+h/2} \frac{e^{-\gamma z}}{z^3}\, dz\right] \tag{30}$$

Let us now compute (30). The integrals I_2 and I_3 can be found in any standard table of integrals. They are given by:

$$I_2(x) = \int \frac{e^{ax}}{x^2}\, dx = -\frac{e^{ax}}{x} + a\int \frac{e^{ax}}{x}\, dx \tag{31.a}$$

$$I_3(x) = \int \frac{e^{ax}}{x^3}\, dx = -\frac{e^{ax}}{2x^2} - \frac{a\, e^{ax}}{2x} + \frac{a^2}{2}\int \frac{e^{ax}}{x}\, dx \tag{31.b}$$

where a new integral must be computed, and can also be found in tables. It is given by:

$$I_1(x) = \int \frac{e^{ax}}{x}\, dx = \ln|x| + a\,x + \frac{a^2 x^2}{2\ \ 2!} + \frac{a^3 x^3}{3\ \ 3!} + \ \ldots \tag{31.c}$$

By using (31.a) and (31.b) in (30) and arranging terms we obtain:

$$\Delta R(z)\big|_{z\to\infty} \approx \frac{h}{2\pi\sigma}\left[-\frac{\gamma\, e^{-\gamma z}}{2\,z} - \frac{e^{-\gamma z}}{2\,z^2} - \frac{\gamma^2}{2}\, I_1(z)\right]_{z-h/2}^{z+h/2} \tag{32}$$

where we are going to refer to the three terms inside the brackets, from left to right, as A(z), B(z) and C(z).

Let us consider each term separately. Starting with A(z) we have that:

$$A(z) = -\frac{\gamma\, e^{-\gamma z}}{2\,z}\bigg|_{z-h/2}^{z+h/2} = -\frac{\gamma}{2}\left[\frac{e^{-\gamma(z+h/2)}}{z+h/2} - \frac{e^{-\gamma(z-h/2)}}{z-h/2}\right] \tag{33.a}$$

and after some algebraic manipulations:

$$A(z) = \frac{\gamma\, e^{-\gamma z}}{z^2 - h^2/4}\left[z\,\mathrm{Sinh}(\gamma\, h/2) + \frac{h}{2}\,\mathrm{Cosh}(\gamma\, h/2)\right] \tag{33.b}$$

where some approximations can be done. As we are considering z large, it is reasonable to assume that z >> h, by doing so we can say that:

$$z^2 - h^2/4 \approx z^2 \tag{33.c}$$

Also, let us assume that $\gamma h < 1$, which can be reasonable for a broad range of frequencies. By doing so, we can say that:

$$\mathrm{Sinh}(\gamma\, h/2) \approx \gamma\, h/2 \quad \text{and} \quad \mathrm{Cosh}(\gamma\, h/2) \approx 1 \tag{33.d}$$

Then, by applying (33.c) and (33.d) to (33.b), we get:

$$A(z)\big|_{z\to\infty} \approx e^{-\gamma z}\left[\frac{\gamma^2 h}{2z} + \frac{\gamma\, h}{2\,z^2}\right] \tag{33.e}$$

Next, let us compute the second term, which is given by:

$$B(z) = -\frac{e^{-\gamma z}}{2\,z^2}\bigg|_{z-h/2}^{z+h/2} = -\frac{1}{2}\left[\frac{e^{-\gamma(z+h/2)}}{(z+h/2)^2} - \frac{e^{-\gamma(z-h/2)}}{(z-h/2)^2}\right] \tag{34.a}$$

and after some algebraic manipulations:

$$B(z) = \frac{e^{-\gamma z}}{(z^2 - h^2/4)^2}\left[z^2\,\mathrm{Sinh}(\gamma\, h/2) + zh\,\mathrm{Cosh}(\gamma\, h/2) + \frac{h^2}{4}\,\mathrm{Sinh}(\gamma h/2)\right] \tag{34.b}$$

By making again the same approximations given by (33.c) and (33.d), B(z) reduces to:

$$B(z)\Big|_{z\to\infty} \approx e^{-\gamma z}\left[\frac{\gamma h}{2z^2} + \frac{h}{z^3} + \frac{\gamma h^3}{8z^4}\right] \tag{34.c}$$

The last term in (32) is given by:

$$C(z) = -\frac{\gamma^2}{2} I_1(z)\Big|_{z-h/2}^{z+h/2} = -\frac{\gamma^2}{2}\int_{z-h/2}^{z+h/2}\frac{e^{-\gamma z}}{z}\,dz \tag{35.a}$$

where after applying (31.c), solving some powers and organizing terms, the following infinite series is obtained:

$$C(z) = -\frac{\gamma^2}{2}\left[\ln\left|\frac{z+h/2}{z-h/2}\right| - \gamma h + \frac{\gamma^2 z h}{2!} - \frac{\gamma^3 z^2 h}{3!} - \frac{\gamma^3 h}{12\;\;3!} + \frac{\gamma^4 z^3 h}{4!} + \frac{\gamma^4 z h^3}{4\;\;4!} - \frac{\gamma^5 z^4 h}{5!} - \ldots\right] \tag{35.b}$$

Two approximations can be performed to (35.b). First, let us retain only the leading terms for every power of γ, which are defined by the highest powers of z. Again, this is justified because z is supposed to be large. In this way, (35.b) becomes:

$$C(z)\Big|_{z\to\infty} \approx -\frac{\gamma^2}{2}\left[\ln\left|\frac{z+h/2}{z-h/2}\right| - \gamma h + \frac{\gamma^2 z h}{2!} - \frac{\gamma^3 z^2 h}{3!} + \frac{\gamma^4 z^3 h}{4!} - \frac{\gamma^5 z^4 h}{5!} + \ldots\right] \tag{35.c}$$

For the second approximation, let us consider the following series expansion for the logarithm:

$$\ln y = \frac{y-1}{y} + \frac{1}{2}\left(\frac{y-1}{y}\right)^2 + \ldots \quad \text{for } y > 1/2 \tag{35.d}$$

where, in this particular case, y = (z+h/2) / (z-h/2). Notice that, because of z >> h, the value of y will always be very close to 1; for this reason, we can use only the first term in (35.d) for approximating the logarithm in (35.c). Then, we will have:

$$\ln\left|\frac{z+h/2}{z-h/2}\right|\Bigg|_{z\to\infty} \approx \frac{\dfrac{z+h/2}{z-h/2} - 1}{\dfrac{z+h/2}{z-h/2}} = \frac{z+h/2-z+h/2}{z+h/2} = \frac{h}{z+h/2} \tag{35.e}$$

By using again the fact that z >> h, it is possible to simplify (35.e) further more. By doing so, (35.e) reduces to:

$$\ln\left|\frac{z+h/2}{z-h/2}\right|\Bigg|_{z\to\infty} \approx \frac{h}{z+h/2} \approx \frac{h}{z} \tag{35.f}$$

Now, by replacing (35.f) into (35.c) we obtain:

$$C(z)\Big|_{z\to\infty} \approx -\frac{\gamma^2}{2}\left[\frac{h}{z} - \frac{\gamma h}{1!} + \frac{\gamma^2 z h}{2!} - \frac{\gamma^3 z^2 h}{3!} + \ldots\right] = -\frac{\gamma^2 h}{2z}\left[1 - \frac{\gamma z}{1!} + \frac{\gamma^2 z^2}{2!} - \frac{\gamma^3 z^3}{3!} + \ldots\right] \qquad (35.g)$$

where the series in brackets is clearly the exponential function. In this way, C(z) is given by:

$$C(z)\Big|_{z\to\infty} \approx -\frac{\gamma^2 h}{2z}\, e^{-\gamma z} \qquad (35.h)$$

Finally, let us replace (33.e), (34.c) and (35.h) into (32) in order to get the asymptotic expansion of ΔR(z) we wanted to compute. After rearranging terms:

$$\Delta R(z)\Big|_{z\to\infty} \approx \frac{h^2}{2\pi\sigma}\left[\frac{\gamma}{z^2} + \frac{1}{z^3} + \frac{\gamma h^2}{8z^4}\right] e^{-\gamma z} \qquad (36)$$

Notice that in the D.C. case $\gamma = 0$ and (36) reduces to:

$$\Delta R(z)\Big|_{z\to\infty} \approx \frac{h^2 z^{-3}}{2\pi\sigma} \qquad (37)$$

which, except for the sign, is the same result obtained in (20) by using the Mellin Transform Method.

CONCLUSIONS

Although we actually found an asymptotic expansion for the potential difference integral, it is only valid for the homogeneous problem. It seems to be no way for computing an expansion for the general non-homogeneous case. For this reason, the use of asymptotic approximations cannot be considered as an alternative for solving the time harmonic field electric logging problem.

Nevertheless, the knowledge of (36) is of great importance for understanding the physics and the properties of the time harmonic field electric logging problem. Also, (36) can be used as a reference for validating the results of the electric logging algorithms when considering an homogeneous earthen formation.

REFERENCES

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