Integration in the Complex λ Plane

Rafael E. Banchs

INTRODUCTION

The purpose of this report is to present a detailed analysis of the methodology used in an attempt to solve the integrals in the electric logging problem by using complex variable theory. Although this methodology happened to be theoretically suitable, in practice the behavior of the integrals and the residues when the operation frequency approaches the DC limit makes the method numerically irresolvable.

INTEGRAL FOR COMPUTING THE POTENTIAL DIFFERENCES

Figure 1 presents a current element in a conductive earthen formation composed by coaxial annular zones of constant electric parameters.

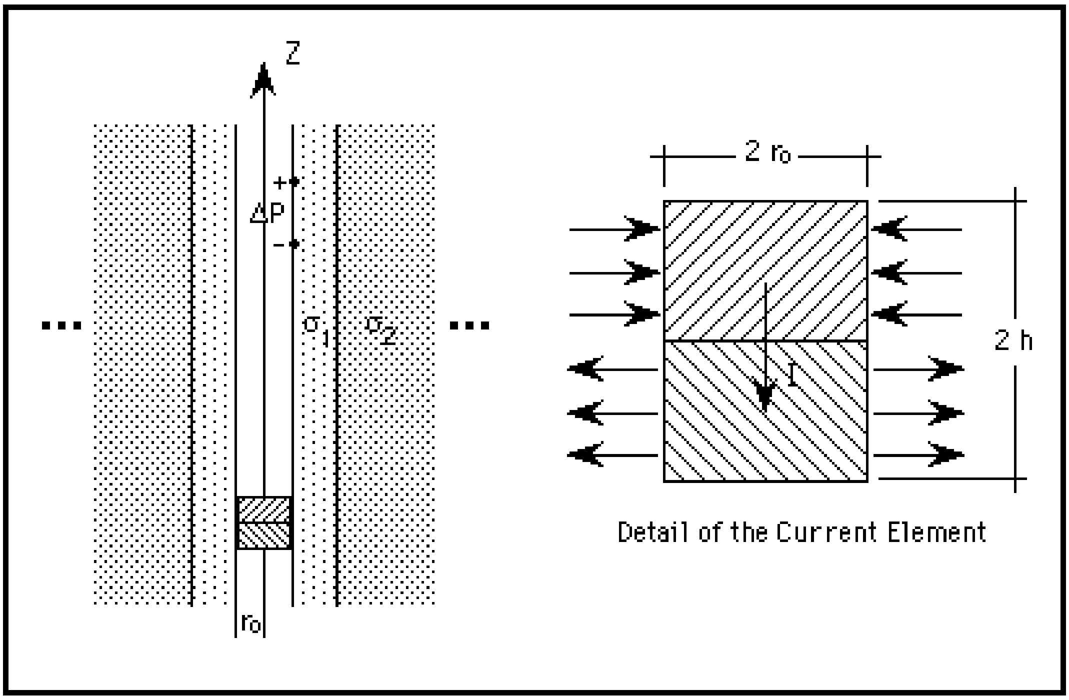

Figure 1: Current element and potential difference measurement.

We are interested in computing the potential difference ΔP(z), generated by the current element, between the points (r_0,z+h/2) and (r_0,z-h/2) as illustrated in Figure 1. It is assumed that the current element and the formation present cylindrical geometry around the z axis. It can be verified from [1] that the potential difference ΔP(z), due to an element located at z = 0, is given by:

$$\Delta P(z) = \frac{2I}{h r_0 \sigma_1 \pi^2} \int_{-\infty}^{\infty} ß_1 \frac{K_0(ß_1 r_0)}{K_0^{'}(ß_1 r_0)} \, \frac{\mathrm{Sin}^3(\lambda h/2)}{\lambda^3} \, e^{-j\lambda z} \, d\lambda \qquad (1)$$

where $ß_1 = ß_1(\lambda) = \pm\sqrt{\lambda^2 + \gamma_1{}^2} = \pm\sqrt{\lambda^2 + j\omega\mu(\sigma_1 + j\omega\varepsilon)}$, r_0 is the radius of the logging tool, h is the segment length, ω is the tool angular frequency of operation, σ_1 is the conductivity of the zone 1, I is the strength of the current element and K_0 is the Modified Bessel function of second kind and order zero.

In (1), the function $K_0(ß_1 r_0)/K_0^{'}(ß_1 r_0)$ stands for the homogeneous problem ($\sigma_2 = \sigma_1$ in Figure 1). In general [1], it takes the form $[K_0(ß_1 r_0) + \Gamma_1 I_0(ß_1 r_0)]/[K_0^{'}(ß_1 r_0) + \Gamma_1 I_0^{'}(ß_1 r_0)]$; where I_0 is the modified bessel function of first kind and order zero, and Γ_1 is the reflection factor in zone 1. Let's denote this function as BR(ß,Γ) or simply BR(ß) in the homogeneous case.

The numerical solution of (1) presents some difficulties because of the particular behavior of the integrand. Some simple and more involving integration methods have been tested with poor results. In the following sections the solution of (1) by using complex variable theory is presented.

THE COMPLEX λ PLANE

Consider the multivalued function:

$$\beta(\lambda) = \pm\sqrt{\lambda^2 + \gamma^2} = \pm\sqrt{\lambda^2 + j\omega\mu(\sigma + j\omega\varepsilon)} \quad (2)$$

Let us call λ_B the branch points of the function (the points where ß(λ) is univalued [2]). These points are given by those values of λ for which the argument of the square root in (2) is equal to zero. By factoring and equating to zero that argument, λ_B can be computed as follows:

$$\lambda^2 + j\omega\mu(\sigma + j\omega\varepsilon) = \left[\lambda + \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}\right]\left[\lambda - \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}\right] = 0 \quad (3)$$

$$\Rightarrow \lambda_B = \pm\sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)} \quad (4.a)$$

or, by solving the square root:

$$\lambda_B = \pm\left[\sqrt{\frac{\omega\mu}{2}\left(\sqrt{\omega^2\varepsilon^2 + \sigma^2} + \omega\varepsilon\right)} - j\sqrt{\frac{\omega\mu}{2}\left(\sqrt{\omega^2\varepsilon^2 + \sigma^2} - \omega\varepsilon\right)}\right] \quad (4.b)$$

where it becomes clear that the function in (2) has two branch points. Let's call them λ_{B1} (when the plus sign is used) and λ_{B2} (when the minus sign is used). Then, ß(λ_{B1}) = ß(λ_{B2}) = 0.

Now, we have to define the branch cuts in order to choose the integration paths properly and assure we are performing all the integration on the same Riemann Sheet [2]. To do that, let us rewrite (3) in terms of the branch points as follows:

$$\beta^2(\lambda) = \beta_1\ \beta_2 = |\beta_1||\beta_2|\exp(j(\alpha_1 + \alpha_2)) \quad (5)$$

where:

$$\beta_1 = \beta_1(\lambda) = |\beta_1| \exp(j\alpha_1) = \lambda - \lambda_{B1} \quad (6.a)$$

$$\beta_2 = \beta_2(\lambda) = |\beta_2| \exp(j\alpha_2) = \lambda - \lambda_{B2} \quad (6.b)$$

so that the function ß(λ) can be expressed as:

$$\beta(\lambda) = \sqrt{|\beta_1||\beta_2|} \exp\left(j\frac{\alpha_1 + \alpha_2}{2}\right) = |\beta| \exp(j\,\alpha) \quad (7)$$

From (5), it follows that in order to stay on the same Riemann Sheet, the phase of ß^2(λ) must be between π and -π; and then from (7), it follows that the phase of ß(λ) should be restricted between π/2 and -π/2. This condition can be achieved by locating the branch cuts in the geometric place:

$$\mathrm{Imag}[ß^2(\lambda)] = 0 \quad (8)$$

By defining the new variables λ_R and λ_I as the real and imaginary parts of λ, we can rewrite (3) as:

$$\beta^2(\lambda) = \lambda^2 + j\omega\mu(\sigma + j\omega\varepsilon) = \lambda_R{}^2 - \lambda_I{}^2 - \omega^2\mu\varepsilon + j(2\lambda_R\lambda_I + \omega\mu\sigma) \quad (9)$$

and by applying (8) to (9) the geometric place mentioned above can be computed. It is given by the following hyperbola:

$$\lambda_R = \frac{-\omega\mu\sigma}{2\lambda_I} \quad (10)$$

Notice that this hyperbolic path includes the branch points λ_{B1} and λ_{B2} .

By using (8) and (2), an expression for ß(λ) along the branch cuts is obtained, and it is given by:

$$\beta(\lambda) = \pm\sqrt{\lambda_R{}^2 - \lambda_I{}^2 - \omega^2\mu\varepsilon} = \pm K|\beta| \quad (11)$$

where λ_R and λ_I are related as in (10) and K can be either 1 or j depending if the amount inside the square root is positive or negative. Then, in order to satisfy the previously stated condition, the branch cuts have to be chosen in such a way that along them ß(λ) equates $\pm$ j |ß|, that is where the amount inside the square root in (11) is negative. Figure 2 illustrates the location of the branch cuts on the complex λ plane and the phase of ß(λ) along them.

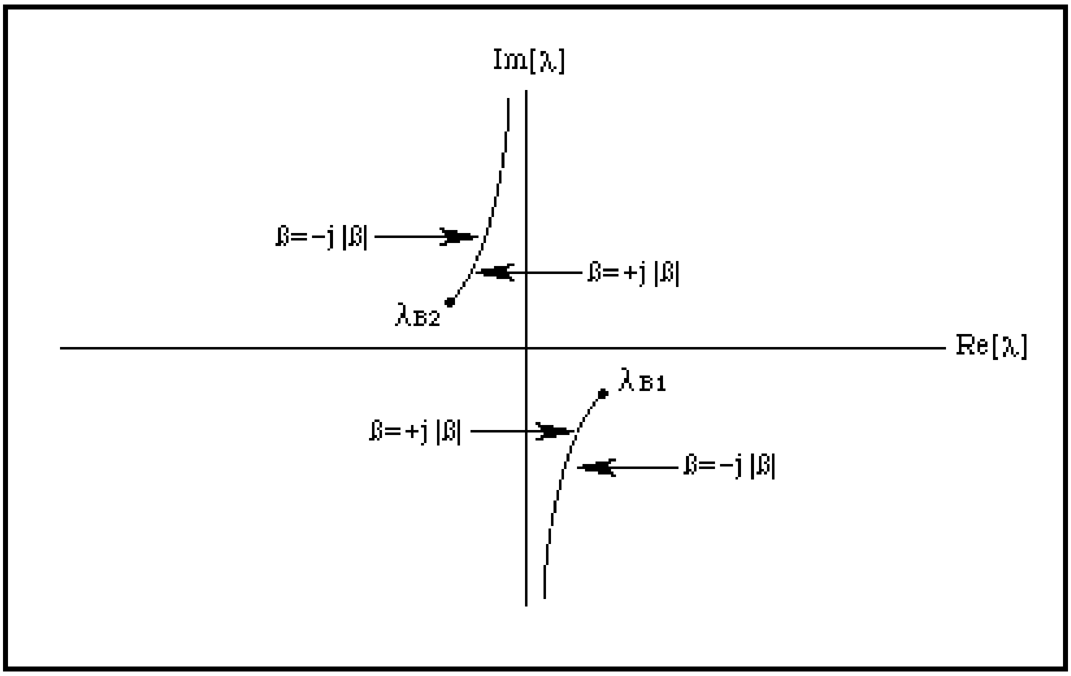

Figure 2: Location of the branch points and the branch cuts in the complex λ plane.

Now we can define the complex integration paths and the integrals that have to be solved in order to compute the integral of interest (along the real axis). Figure 3 shows two closed integration paths (one on the upper semiplane and the other in the lower semiplane) and the integrals associated with them.

By applying the Cauchy theorem to each of the two closed paths in Figure 3 we obtain:

for Path1 (lower semiplane): $I + I_A + I_1^- + I_1^+ + I_B = -2\pi j \sum \text{residues in region 1}$ (12.a)

and for Path2 (upper semiplane): $I + I_C + I_2^- + I_2^+ + I_D = 2\pi j \sum \text{residues in region 2}$ (12.b)

where I is the integral along the real axis we are interested in.

The use of either Path1 or Path2 depends on the convergence of the integrals associated with the paths. If convergence is not achieved in any of those paths, the function to be integrated must be decomposed in terms in such a way that convergence is achieved when integrating each of the terms separately along one of the two paths. In all the cases, the decomposition must be done such that either I_A and I_B or I_C and I_D vanish.

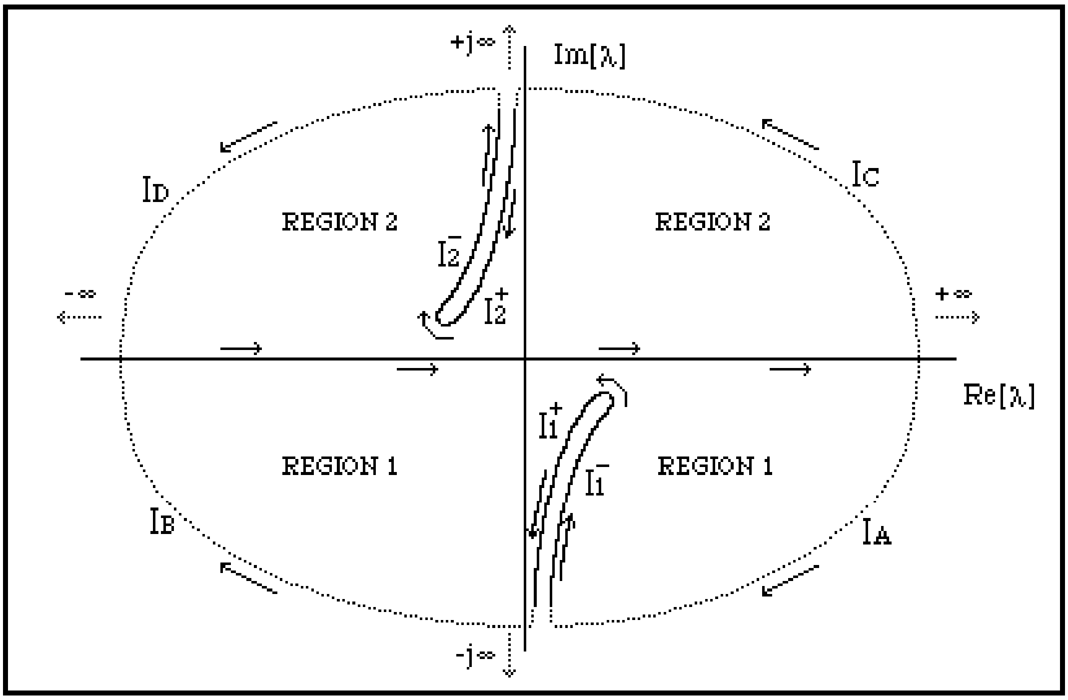

Figure 3: Integration paths and integrals.

THE POTENTIAL DIFFERENCE INTEGRAL IN THE COMPLEX PLANE

It is clear at this moment that due to the sine cube factor, the integral in (1) will not converge neither in Path 2 nor in Path 1 of Figure 3. This is because the sine becomes unbounded when the imaginary part of λ goes to ∞ or -∞. For this reason it is required to decompose the sine function into a sum of complex exponentials and consider each case in a separate integral.

By using the following trigonometric identity:

$$\mathrm{Sin}^3\theta = \frac{1}{4}\left[3\mathrm{Sin}\theta - \mathrm{Sin}3\theta\right] = \frac{1}{8j}\left[3(e^{j\theta} - e^{-j\theta}) - (e^{j3\theta} - e^{-j3\theta})\right] \qquad (13)$$

(1) can be rewritten as a sum of four integrals as follows:

$$\Delta P(z) = \frac{I}{4\, j\, h\, r_0 \sigma_1 \pi^2} \left[3 I_1(z) - 3 I_2(z) - I_3(z) + I_4(z) \right] \tag{14}$$

where: $I_1(z) = \int_{-\infty}^{\infty} \frac{\beta_1 BR(\beta_1)}{\lambda^3} \; e^{-j\lambda(z-h/2)} \, d\lambda$

$$I_2(z) = \int_{-\infty}^{\infty} \frac{\beta_1 BR(\beta_1)}{\lambda^3} \; e^{-j\lambda(z+h/2)} \, d\lambda$$

$$I_3(z) = \int_{-\infty}^{\infty} \frac{\beta_1 BR(\beta_1)}{\lambda^3} \; e^{-j\lambda(z-3h/2)} \, d\lambda$$

$$I_4(z) = \int_{-\infty}^{\infty} \frac{\beta_1 BR(\beta_1)}{\lambda^3} \; e^{-j\lambda(z+3h/2)} \, d\lambda$$

Notice that I_1, I_2 , I_3 and I_4 are integrals of the general form:

$$I(z) = \int_C \frac{f(\lambda)}{\lambda^3} \; e^{-j\lambda(z+k)} \, d\lambda \qquad \text{for } k = -3h/2,\ -h/2,\ h/2,\ 3h/2 \tag{15}$$

Two important considerations have to be done for this kind of integral:

1.- There is a triple pole at $\lambda=0$, so its residue must be computed.

2.- The integration path must be chosen so that the integral converges.

The residue can be computed as:

$$\text{residue} = R = \frac{1}{2!} \frac{d^2}{d\lambda^2} \left[f(\lambda)\, e^{-j\lambda(z+k)} \right]_{\lambda \to 0} \tag{16}$$

and it can be verified that for the homogeneous problem, it is given by:

$$R = -\frac{1}{2} \left[\left(\frac{K_0^{\,2}(\gamma r_0)}{K_1^{\,2}(\gamma r_0)} - 1 \right) r_0 + \left(\frac{2}{\gamma} - \gamma (z+k)^2 \right) \frac{K_0(\gamma r_0)}{K_1(\gamma r_0)} \right] \tag{17}$$

where $\gamma^2 = j\omega\mu\,(\sigma + j\omega\varepsilon)$

Notice that when ω -> 0 then $|R|$ -> ∞. This fact constitutes the main limitation of this method.

Now let us consider the convergence of the integral. In order to achieve it, the real part of the argument of the exponential function must be negative. That argument can be separated into real and imaginary parts as follows:

$$-j\lambda(z+k) = \lambda_i(z+k) - j\lambda_r(z+k) \qquad (18)$$

where λ_i and λ_r are the imaginary and real parts of λ respectively. Notice that when $(z+k) > 0$, λ_i must be negative to achieve convergence, while when $(z+k) < 0$, λ_i must be positive. This provides the criteria for choosing between Path 1 and Path 2 in (12). This consideration not only ensures the convergence of the integrals, but also that I_A, I_B, I_C and I_D in (12.a) and (12.b) vanish. The following table presents the paths that must be used to computed the integrals I_1, I_2 , I_3 and I_4 in (14) for different intervals of z.

Table #1: Integration paths to be used for the integrals in different intervals of z.

Integral	Value of K	z < -3h/2	(-3h/2 : -h/2)	(-h/2 : h/2)	(h/2 : 3h/2)	z > 3h/2
I_1	K = -h/2	Path 2	Path 2	Path 2	Path 1	Path 1
I_2	K = h/2	Path 2	Path 2	Path 1	Path 1	Path 1
I_3	K = -3h/2	Path 2	Path 2	Path 2	Path 2	Path 1
I_4	K= 3h/2	Path 2	Path 1	Path 1	Path 1	Path 1

There is still a final consideration that must be done. That is the fact that the poles are in the origin and both Path 1 and Path 2 pass through the origin. The solution of this problem is very simple, we just have to assume that the contour that follows the real axis passes below or above the singularity at $\lambda = 0$ and therefore the residue will only contribute to the integral in the upper or lower semiplane respectively. By assuming the contour passes below the singularity and following the integration criteria depicted in Table 1, equations (12.a) and (12.b) reduce to:

for Path 1: $I = -I_1^- - I_1^+$ (19.a)

and for Path 2: $I = 2\pi jR - I_2^- - I_2^+$ (19.b)

where I refers to any of the four integrals in (14); R is the residue, which is given by (17) for the homogeneous case; and the remaining integrals are those along the branch cuts that were illustrated in Figure 3.

COMPUTATION OF THE INTEGRALS

Finally, let us compute $I_1^+, I_1^-, I_2^+ \text{and} I_2^-$. In fact some simplifications can be done due to the similitude among the integrals.

First let us consider I_1^+:

$$I_1^+(z) = \int_{C_1^+} F^+(\lambda)\, e^{-j\lambda(z+k)}\, d\lambda \tag{20}$$

where $F^+(\lambda) = \dfrac{\beta_1}{\lambda^3}\, BR(\beta_1 r_0) = \dfrac{j|\beta_1|}{\lambda^3}\, BR(j|\beta_1| r_0)$ (21)

Notice that (21) is true only for those values of λ along the hyperbolic paths (branch cuts) described by (10). Also, remember that the sign of $ß_1$ must be chosen in order to satisfy the phase condition discussed two sections before. Figure 2 shows the appropriated sign for each of the integrals.

By using (10) it is possible to express λ along the branch cuts as a function of its own imaginary part as follows:

$$\lambda(\lambda_i) = \frac{-\omega\mu\sigma}{2\lambda_i} + j\lambda_i \tag{22.a}$$

and, $\dfrac{d\lambda}{d\lambda_i} = \dfrac{\omega\mu\sigma}{2\lambda_i^2} + j$ (22.b)

so (20) can be rewritten as:

$$I_1^+(z) = \int_{\lambda_{B1}^i}^{-\infty} F^+(\lambda(\lambda_i)) \left(\frac{\omega\mu\sigma}{2\lambda_i^2} + j\right) e^{j\frac{\omega\mu\sigma}{2\lambda_i}(z+k)}\, e^{\lambda_i(z+k)}\, d\lambda_i \tag{23}$$

where λ_{B1}^i is the imaginary part of the branch point λ_{B1}, which is given in (4.b).

Proceeding in a similar way I_1^- can be computed. It is given by:

$$I_1^-(z) = \int_{-\infty}^{\lambda_{B1}^i} F^-(\lambda(\lambda_i)) \left(\frac{\omega\mu\sigma}{2\lambda_i^2} + j\right) e^{j\frac{\omega\mu\sigma}{2\lambda_i}(z+k)}\, e^{\lambda_i(z+k)}\, d\lambda_i \tag{24}$$

where $F^{-}(\lambda) = \frac{\beta_1}{\lambda^3} BR(\beta_1 r_0) = \frac{-j|\beta_1|}{\lambda^3} BR(-j|\beta_1| r_0)$ (25)

By substituting (23) and (24) in (19.a) and combining the integrals we get:

$$I = -I_1^- - I_1^+ = \int_{\lambda_{B1}^i}^{-\infty} F(\lambda(\lambda_i)) \left(\frac{\omega\mu\sigma}{2\lambda_i^2} + j \right) e^{j\frac{\omega\mu\sigma}{2\lambda_i}(z+k)} e^{\lambda_i (z+k)} d\lambda_i \qquad (26)$$

where $F(\lambda(\lambda_i)) = F^{-}(\lambda(\lambda_i)) - F^{+}(\lambda(\lambda_i))$ (27)

By applying a similar analysis, (19.b) can be rewritten as follows:

$$I = 2\pi jR - I_2^- - I_2^+ = 2\pi jR - \int_{\lambda_{B2}^i}^{\infty} F(\lambda(\lambda_i)) \left(\frac{\omega\mu\sigma}{2\lambda_i^2} + j \right) e^{j\frac{\omega\mu\sigma}{2\lambda_i}(z+k)} e^{\lambda_i (z+k)} d\lambda_i \qquad (28)$$

where $F(\lambda(\lambda_i))$ is the same as in (27).

A final simplification can be done by remembering that in (28) it will always hold that (z+k) < 0. Remember that this was the convergence criteria used to generate Table 1. Therefore, in (28) it will always be true that (z+k) = - | z+k |; while in (26) it will be true that (z+k) = | z+k |. By using this properties and making the change of variables $\lambda_i = -\lambda_i$ into (28) it can be verified that:

$$-I_1^- - I_1^+ = -\left(-I_2^- - I_2^+\right) \qquad (29)$$

Finally, by using (29), equations (26) and (28) can be combined into a single equation as follows:

$$I(z) = n\, 2\pi jR + (-1)^n \int_{\lambda_{B1}^i}^{-\infty} F(\lambda(\lambda_i)) \left(\frac{\omega\mu\sigma}{2\lambda_i^2} + j \right) e^{j\frac{\omega\mu\sigma}{2\lambda_i}|z+k|} e^{\lambda_i |z+k|} d\lambda_i \qquad (30)$$

where n must be set to 0 if we want to evaluate equation (26) or to 1 for evaluating (28). The criteria for setting n is the same criteria to select the paths described in Table 1; n=0 corresponds to Path 1 and n=1 to Path 2.

The numerical evaluation of (30) should not present major difficulties because of the presence of the real exponential. A conventional numerical integration technique must be suitable [3].

To compute the potential difference $\Delta P(z)$, (30) must be solved for the four values of k presented in (15) and the four results must be combined according to (14).

CONCLUSIONS

As it was noticed in (17), the value of the residue tends to infinity when the frequency of operation approaches to zero. This problem can be solved for the DC limit, but it will always be present for small values of $\omega > 0$. Although in theory, when frequency approaches the DC case, the huge values of the residues cancel with the also huge values of the integrals, in the practice it is impossible to achieve good results due to numerical problems. This fact constitutes the main limitation of this methodology and makes its implementation worthless.

Another limitation of this methodology is the computation of the residues for the general case. The residue given by (17) corresponds to the homogeneous problem. The computation of the residue for the non homogeneous problem is not as simple and direct as for the homogeneous. A recursive procedure that allows the computation of the derivatives required to obtain the residues must be developed.

Finally, a numerical problem arises when computing the potential difference at values of z close to those four values of k presented in (15). In these cases the real exponential in (30) varies very slowly and the numerical integration becomes more time consuming.

REFERENCES

[1] Bostick, F.; Smith, H. (1994), Propagation Effects in Electric Logging. University of Texas at Austin.

[2] Moretti, G. (1961), Functions of a Complex Variable. Prentice Hall.

[3] Davis, P.; Rabinowitz, P. (1984), Methods of Numerical Integration. Academic Press.