The Anderson's Integration Technique

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INTRODUCTION

This report describes a numerical integration technique due to Anderson [1]. In this technique a transform-type integral is converted, by an appropriate change of variables, into a convolution-type integral which is finally approximated by a discrete convolution. The main advantage of this technique is given by its computational velocity that is several orders of magnitude faster than conventional integration techniques. However, this integration technique is still an approximation and its performance highly depends on the behavior of the function to be integrated. The best results are obtained for monotonic decreasing continuos integrands [1].

In the solution of the time harmonic electric logging problem, the computation of the ΔR's requires the evaluation of a Cosine Transform integral [2]. The suitability of the Anderson's technique for the numerical computation of this integral is to be evaluated.

THE CHANGE OF VARIABLES

Let us consider the general transform-type integral:

$$I(z) = \int_0^\infty f(\lambda)\ k(z\lambda)\, d\lambda \qquad (1)$$

where f(z) is the function to be integrated and k(z) is the transform kernel.

Now, let us consider the following change of variables:

$$\lambda = e^{-u} \qquad (2.a)$$

$$z = e^{v} \qquad (2.b)$$

so that:

$$d\lambda = -e^{-u}\,du \tag{3}$$

$$u = -\ln(\lambda) \tag{4.a}$$

and $v = \ln(z)$ (4.b)

By applying (2.a), (2.b), (3) and (4) to (1); we get:

$$I(e^{v}) = \int_{\infty}^{-\infty} f(e^{-u})\; k(e^{v-u})\left[-e^{-u}\right]du \tag{5}$$

and after multiplying both sides of (5) by e^{v}, and swapping the limits of integration:

$$e^{v}\, I(e^{v}) = \int_{-\infty}^{\infty} f(e^{-u})\left[e^{v-u}\; k(e^{v-u})\right]du \tag{6}$$

where it can be clearly seen that the integral in (6) is a convolution-type integral.

THE DISCRETE CONVOLUTION

The convolution integral in (6) can be approximated by a discrete convolution. Notice from (6) that the two functions to be convolved can be defined as follows:

$$f_e(v) = f(e^{-v}) \tag{7}$$

$$g_e(v) = e^{v}\, k(e^{v}) \tag{8}$$

where $g_e(v)$ will always be the same for an specific kind of transform. For this reason, it will be called the filter; while $f_e(v)$ will be the input function.

By using the commutative property of the convolution, (6) can be evaluated by shifting the input function instead of the filter. In this way, a discrete version of (6) can be written as:

$$e^{v}\, I(e^{v}) = \sum_{n=-\infty}^{\infty} g_e(x_n)\; f_e(x_n - v) \tag{9}$$

where the x_n's are uniformly spaced samples of the integration variable u in (6).

For practical purposes, the infinite impulse response of the filter in (9) must be approximated by a finite impulse response. As it will be shown later, the impulse response obtained in the design of this kind of filters decays in such a way that it can be truncated without problems.

Finally, a more appropriated expression for the integral I(z) in (1) can be obtained from (9) by using (2.b) and (4.b), expressing the input function in terms of the original function f(λ) and considering a finite impulse response filter of M coefficients. Then, I(z) is given by:

$$I(z) = \frac{1}{z} \sum_{n=1}^{M} g_e(x_n)\, f(e^{(x_n - \ln(z))}) \qquad (10)$$

FILTER DESIGN

As it was mentioned in the introduction, the solution of the time harmonic electric logging problem involves the numerical evaluation of a Cosine Transform integral. Although there exist very good filters available in the literature [1], a brief discussion of the designing technique and a simple design example are presented here for better understanding of the method.

Although it is kind of challenging, the design of this kind of filters is conceptually very simple. Starting from the general transform-type integral presented in (1) and following the same procedure discussed before, equation (6) is obtained. This expression can be rewritten as follows:

$$I_e(v) = f_e(v) * g_e(v) \qquad (11)$$

where $I_e(v)$ is going to be referred as the output function and will be given by:

$$I_e(v) = e^v\, I(e^v)\,, \qquad (12)$$

$f_e(v)$ is going to be referred as the input function and will be given by:

$$f_e(v) = f(e^{-v})\,, \qquad (13)$$

$g_e(v)$ is the filter response we are interested in designing and the symbol ***** denotes convolution.

By Fourier transforming both sides of (11) and using the convolution property of the Fourier Transform, we get:

$$\Im[I_e(v)] = \Im[f_e(v)]\ \Im[g_e(v)] \tag{14}$$

where $\Im[\ ]$ is the Fourier Transform operator.

For a known input-output pair of functions, a filter can be designed by solving (14) for $\Im[g_e(v)]$ and taking the Inverse Fourier Transform. Then, the impulse response of $g_e(v)$ is given by:

$$g_e(v) = \Im^{-1}\left[\frac{\Im[I_e(v)]}{\Im[f_e(v)]}\right] \tag{15}$$

Notice from (15) that special care must be taken to avoid division by zero. This is usually achieved by selecting appropriate input and output functions.

DESIGN EXAMPLE

Let us consider the Cosine Transform integral defined by:

$$I(z) = \int_0^\infty f(\lambda)\, \mathrm{Cos}(z\lambda)\, d\lambda \tag{16}$$

First of all, we have to find a transformed pair of functions I(z) and f(λ) such that satisfy (16). An appropriated pair is given by:

$$I(z) = \frac{1}{2}\exp\left(\frac{-z^2}{4\pi}\right) \tag{17}$$

$$f(\lambda) = \exp(-\pi\lambda^2) \tag{18}$$

Using (17) and (18) does not always yields to good filter responses. It was found by Anderson that the filter responses are improved when rapidly decreasing input and output functions are used in the design [1]. For this reason, it is better to construct a more suitable transform pair rather than (17) and (18). Starting from (17) and (18), and using the linearity of the Cosine Transform, we can define a new pair of functions:

$$I(z) = \frac{\sqrt{a}}{2}\exp\left(\frac{-a}{4\pi}z^2\right) - \frac{\sqrt{b}}{2}\exp\left(\frac{-b}{4\pi}z^2\right) \tag{19}$$

$$f(\lambda) = \exp\left(\frac{-\pi}{a}\lambda^2\right) - \exp\left(\frac{-\pi}{b}\lambda^2\right) \tag{20}$$

where two independent design parameters, a and b, have been introduced.

Next, a sampling period T_s is chosen and an abscissa vector x_n is computed. So, the output and input functions can be computed by replacing (19) and (20) into (12) and (13) respectively. Discrete output and input sequences are obtained by evaluating (12) and (13) at the abscissa values given by the vector x_n. Then, Discrete Fourier Transform subroutines are used to evaluate (15) and obtain the impulse response of the filter.

In the procedure described by Anderson [1], a predefined sampling period T_s is used and a shift is introduced, as a design parameter, in order to reduce the magnitude of the filter tails. Here, we are going to omit this part of the procedure by using the following simple and intuitive argument. Let us consider the filter response given by (8). Notice that, in the case of an oscillatory kernel, for increasing values of the argument the probability of taking a sample in a zero crossing also increases. Then, it should be easy to get a good design by slightly varying T_s around its desired value. So, in this example, we are going to use the sampling period as the design parameter by itself. For a more rigorous design, refer to the procedure described by Anderson in [1].

Now, let us present a simple design example for a filter of 128 coefficients. The design parameters to be considered were a, b and T_s. The initial value for the sampling period was set to $T_s = 0.5$. Several experiments were run for different values of a and b. The shapes of the Fourier Transform magnitudes and phases where observed until noise was substantially reduced. The values of a = sqrt(2π) and b = sqrt(π/2) happened to be very appropriated. Once a and b were defined, another set of experiments were done in which a reconstruction error was measured for the filters obtained while varying T_s. A minimum of the error was found at $T_s = 0.4358$. Figure 1 presents step by step the computation of the definitive filter. For the definitive filter:

$$a = \sqrt{2\pi};\quad b = \sqrt{\pi/2};\quad T_s = 0.4358;\quad \text{and } x_n = nT_s \quad \text{for } n = -64, -63, \ldots, 62, 63 \tag{21}$$

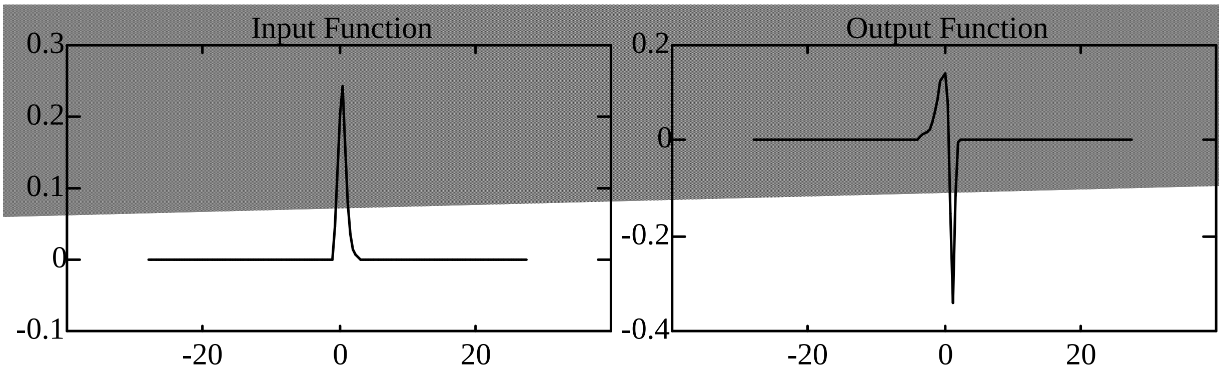

Figure 1.a: Input and Output Functions.

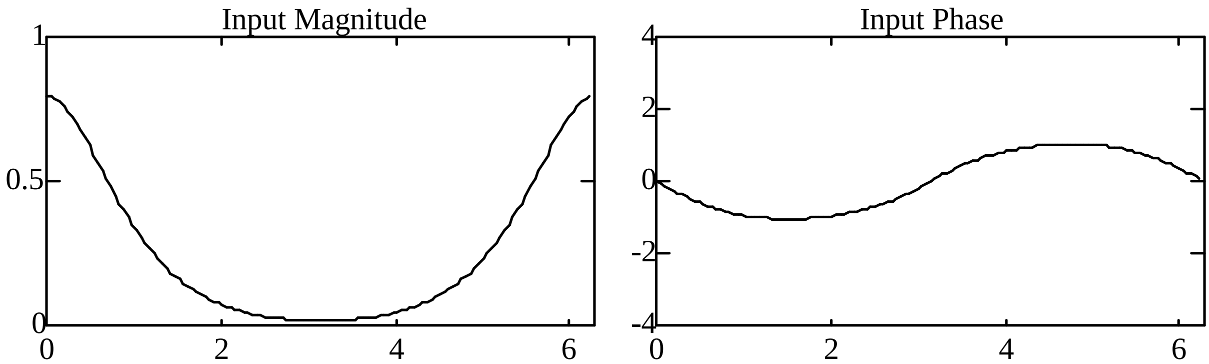

Figure 1.b: Input Function Fourier Transform.

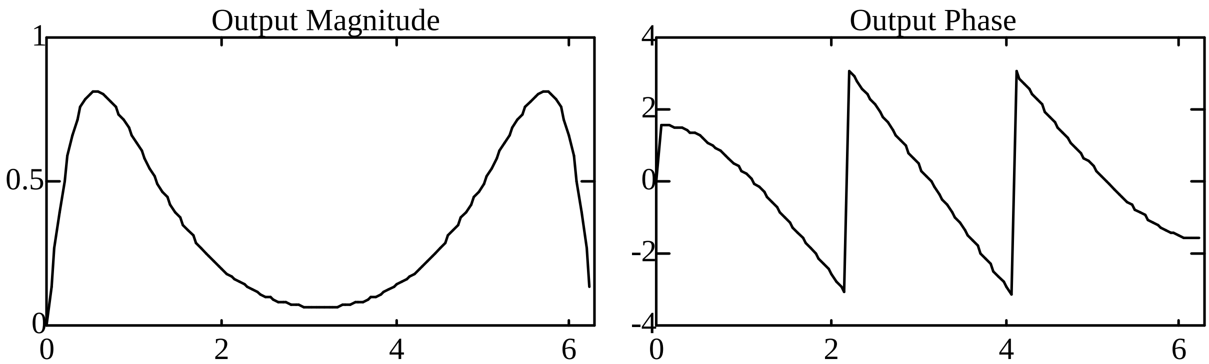

Figure 1.c: Output Function Fourier Transform.

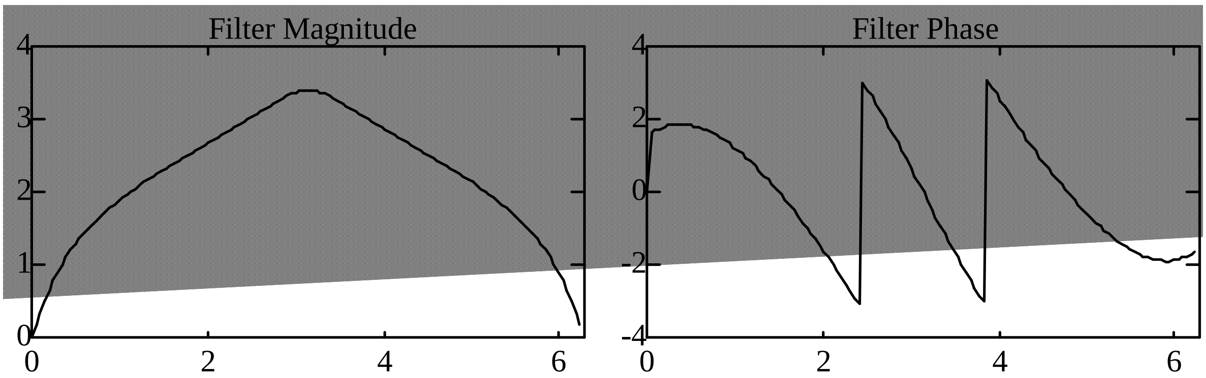

Figure 1.d: Filter Fourier Transform.

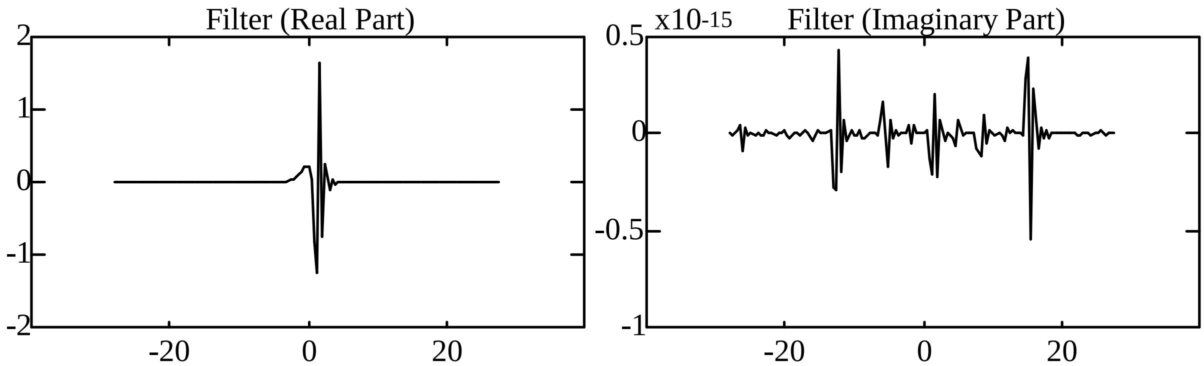

Figure 1.e: Filter Impulse Response.

The imaginary part of the filter impulse response obtained is neglected for obvious reasons.

The filter response obtained in Figure 1.e can be used in (10) for computing an arbitrary Cosine Transform. However, as it was mentioned before, the performance of the filter will depend on the behavior of the function to be integrated. In this particular case, because we used a sampling period of 0.4358, the filter designed has a cutoff frequency of approximately 2.29. So, the filter performance will degrade notoriously if the function to be transformed presents frequency components above that cutoff frequency.

In general, for smaller sampling periods the impulse responses will be better, but the design will become more difficult; while for larger sampling periods the design will be simpler but the impulse response will be poorer. Also, it is generally useful to use a vector x_n larger than the desired impulse response. Then, the filter response can be truncated to the desired length.

FILTER PERFORMANCE

As an example of the performance of the filter designed above, let us consider the following Cosine Transform:

$$I(z) = \int_0^\infty [e^{-\lambda/a} \cos(b\lambda)] \cos(z\lambda)\, d\lambda \qquad (22)$$

which analytic solution is known and is given by:

$$I(z) = \frac{1}{2}\left[\frac{a}{1+a^2(z-b)^2} + \frac{a}{1+a^2(z+b)^2}\right] \qquad (23)$$

By using (10) to evaluate (22) and comparing the result against (23), we can evaluate the performance of the designed filter. Two examples are presented in Figure 2.

Figure 2.a presents the case in which the values of a =1 and b = 0 are considered. Notice than in this case the function to be transformed behaves good in the sense that decreases monotonicaly with no oscillations. The performance for this first case is relatively good. On the other hand, Figure 2.b presents the case in which a =1 and b = 5. In this second case, the function to be transformed presents oscillations which frequency is above the filter cutoff value of 2.29. It is clear that the filter performs very poorly.

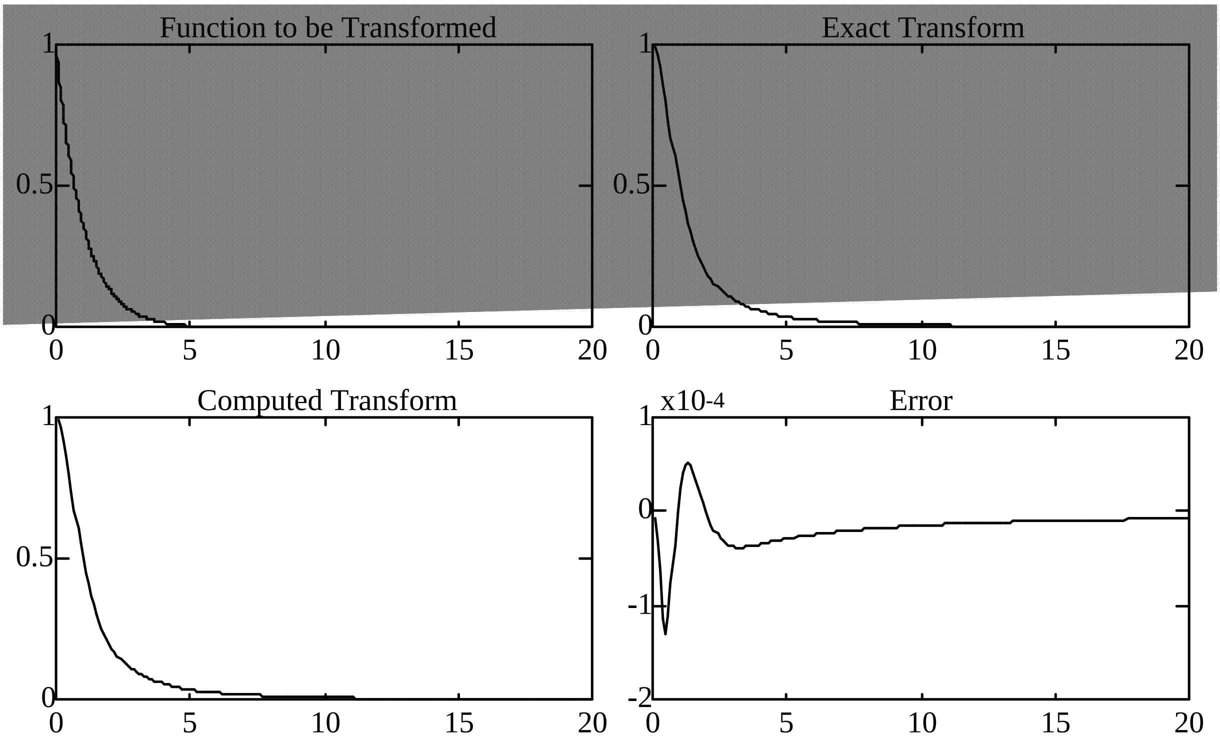

Figure 2.a: Filter Performance when transforming a monotonic function.

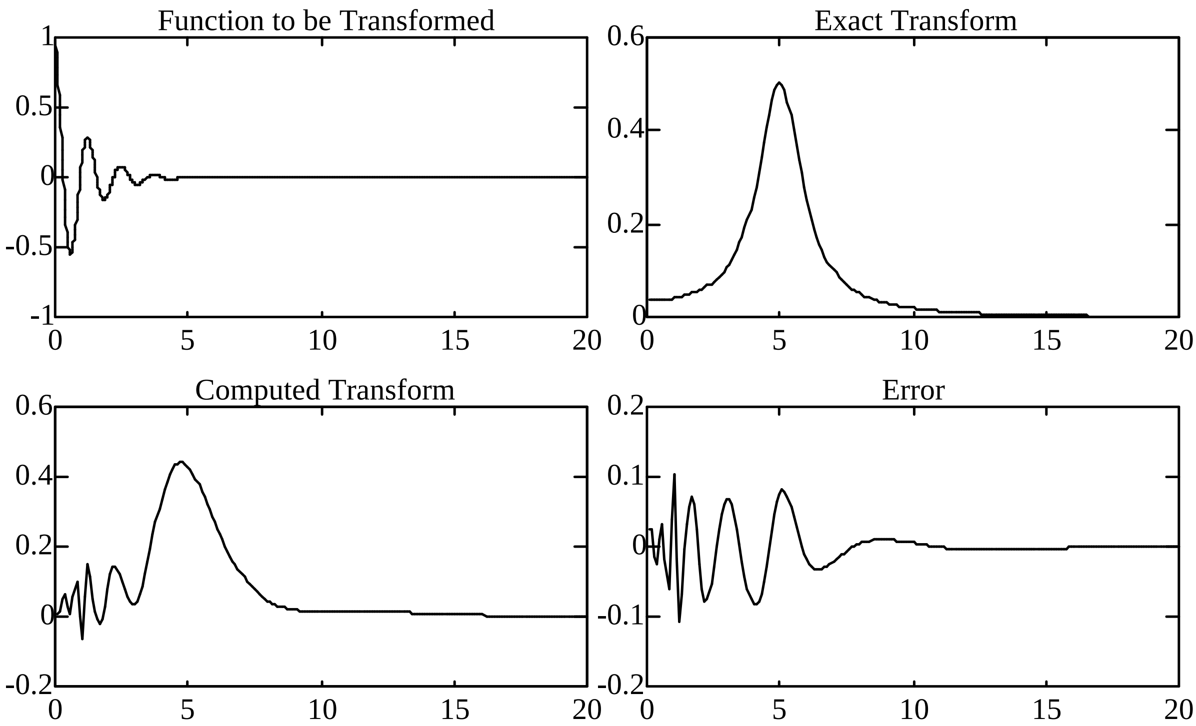

Figure 2.b: Filter Performance when transforming an oscillating function.

THE TIME HARMONIC FIELD ELECTRIC LOGGING PROBLEM

Finally, let us consider the application of this integration technique for the solution of the time harmonic electric logging problem described in [2]. For this purpose, we are going to use a filter designed by Anderson [1] which is much more accurate than the example presented before. The filter we are going to use has 787 coefficients, and a sampling period of 0.1. The integral we are interested in evaluate is presented in [2] and can be written as:

$$\Delta R(z) = \frac{4}{r_0\,\sigma_1\,h\,\pi^2}\int_0^\infty ß_1\left[\frac{K_0(ß_1\,r_0)+\Gamma_1\,I_0(ß_1\,r_0)}{K_0^{'}(ß_1\,r_0)+\Gamma_1\,I_0^{'}(ß_1\,r_0)}\right]\frac{\mathrm{Sin}^3(\lambda h/2)}{\lambda^3}\,\mathrm{Cos}(\lambda z)\,d\lambda \qquad (24)$$

where $ß_1 = ß_1(\lambda) = \pm\sqrt{\lambda^2 + j\omega\mu\,\sigma_1}$, r_0 is the radius of the logging tool, h is the segment length, ω is the angular frequency of operation, σ_1 is the conductivity, Γ_1 is the reflection coefficient and I_0 and K_0 are the zero order Modified Bessel functions of first and second kind.

Figure 3 evaluates the performance of the Anderson's integration technique in the solution of (24). A solution computed by using the Trapezoid Method is presented as a reference. The derivatives of both solutions are also presented.

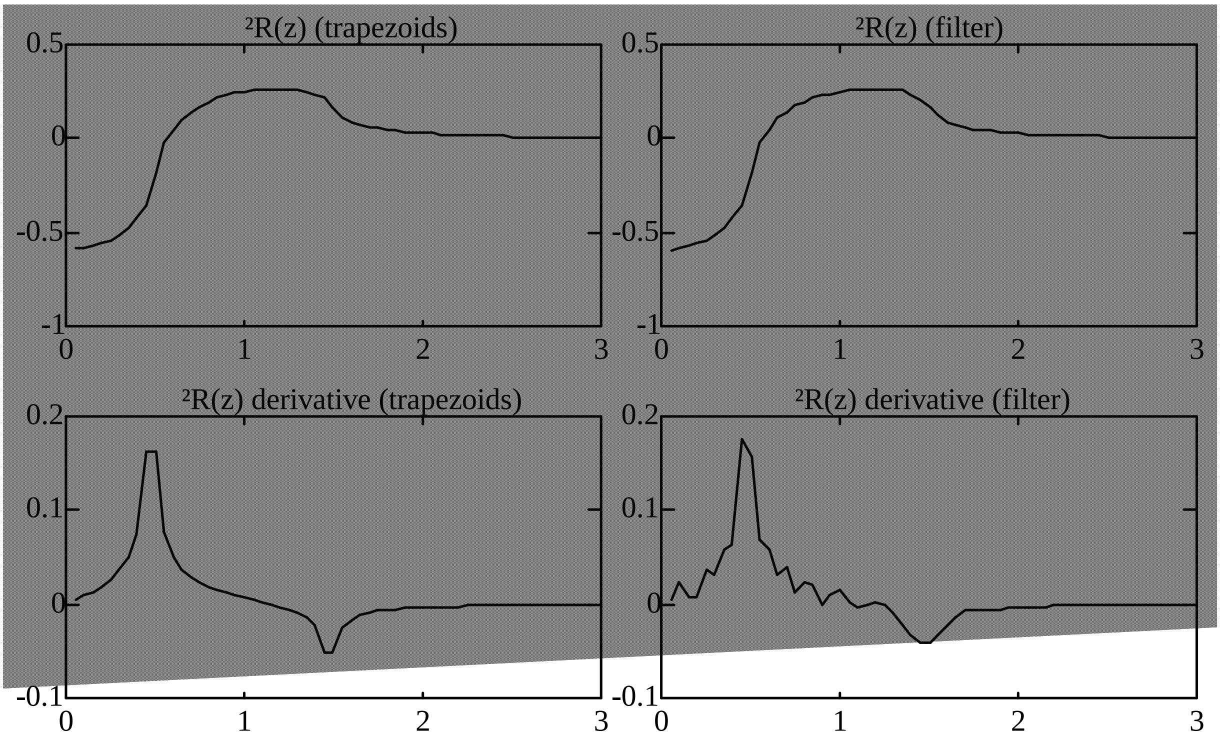

Figure 3: Electric Logging Problem Computation.

CONCLUSIONS

It can be concluded that the Anderson’s integration technique constitutes a very powerful method to evaluate transform-type integrals. Its main advantage is given by its computational speed. However, it still constitutes an approximation and not always its performance is as good as desired.

In the case of the time harmonic field electric logging problem, as it can be seen from Figure 3, the performance of the technique may be considered acceptable, but actually it is not as good as expected. This can be due to the presence of oscillations in the function to be integrated in (24). More alternatives must be evaluated before deciding using this integration technique for solving the time harmonic field electric logging problem.

REFERENCES

[1] Anderson, W. (1975), Improved Digital Filters for Evaluating Fourier and Hankel Transform Integrals. NTIS rep. PB-242-800/1WC, 223p.

[2] Bostick, F.; Smith, H. (1994), Propagation Effects in Electric Logging. University of Texas at Austin.