Hidden Markov Models

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Session 2: Dynamic Programming & HMMs

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Review on HMMs

Last day we defined HMMs, and recalled that they can be formally specified by a set of five elements:

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State Space → $$\{ S, O, \pi, P, Q \}$$

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Observation Space

$$\{ \boldsymbol{S}, \boxed{\boldsymbol{O},} \pi, \mathbf{P}, \mathbf{Q} \}$$

- where, $\boldsymbol{S}$ is the set of states: $\boldsymbol{S} = \{s_1, s_2, s_3 \ldots s_m\}$;
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Model Space: λ

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- ***O*** is the observation alphabet: $\boldsymbol{O} = \{o_1, o_2, o_3 \ldots o_h\}$;
- and, π, **P** and **Q** are the HMM associated probabilities.

Review on HMMs

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- Initial state probabilities: $\pi = \{\pi_i\}$ with $i \in \boldsymbol{S}$
- State transition probabilities: $\mathbf{P} = \{p_{ij}\}$ with $i, j \in \boldsymbol{S}$

- Symbol emission probabilities: $\mathbf{Q} = \{q_{ijk}\}$ with $i, j \in \boldsymbol{S}$, $k \in \boldsymbol{O}$

Review on HMMs

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- **P** is an mxm matrix with elements $p_{ij} = P(X_n = s_j | X_{n-1} = s_i)$ which represent the probability distribution of state transitions.

- **Q** constitutes the observation probability distribution that is associated to state transitions:

$$q_{ijk} = q(Y_n \mid X_n, X_{n-1}) = P(Y_n = o_k \mid X_n = s_j , X_{n-1} = s_i)$$

Probability of an observation sequence

The problem we are going to deal with today is the following:

- Consider a HMM defined as $\lambda = \{\pi, \mathbf{P}, \mathbf{Q}\}$, what is the probability $P(Y|\lambda)$ of a given observation sequence $\boldsymbol{Y} = Y_1, Y_2, Y_3 \ldots Y_N$?

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- $P(Y|\lambda)$ is computed by adding up all probabilities for generating the observation sequence through any possible state sequence.
- Calculating $P(Y|\lambda)$ is very expensive from the computational point of view, and it can be efficiently done by using ***dynamic programming*** techniques.

Mathematical formulation of the problem

$$P(Y\,|\,\lambda) = \sum_{\mathbf{X}} P(Y, X\,|\,\lambda)$$

Mathematical formulation of the problem

observation sequence

given model

$$P(Y\,|\,\lambda) = \sum_{X} P(Y, X\,|\,\lambda)$$

Mathematical formulation of the problem

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joint probability of an observation sequence and a state sequence for the given model

$$P(Y\,|\,\lambda) = \sum_{X} \overbrace{P(Y, X\,|\,\lambda)}$$

all possible state sequences

Mathematical formulation of the problem

observation sequence

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joint probability of an observation sequence and a state sequence for the given model

$$P(Y \mid \lambda) = \sum_{X} \overbrace{P(Y, X \mid \lambda)} = \sum_{X} P(Y \mid X, \lambda)\, P(X \mid \lambda)$$

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Mathematical formulation of the problem

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joint probability of an observation sequence and a state sequence for the given model

$$P(Y\,|\,\lambda) = \sum_{X} \overbrace{P(Y, X\,|\,\lambda)} = \sum_{X} \underbrace{P(Y\,|X, \lambda)}\, P(X\,|\,\lambda)$$

all possible state sequences

observation probability for a given state sequence

Mathematical formulation of the problem

observation sequence

given model

$$P(Y|\lambda) = \sum_{X} \overbrace{P(Y, X|\lambda)}^{\text{joint probability of an observation sequence and a state sequence for the given model}} = \sum_{X} \underbrace{P(Y|X, \lambda)}_{\text{observation probability for a given state sequence}} P(X|\lambda)$$

all possible state sequences

$$P(Y|X, \lambda) = q_{X_0X_1Y_1}\, q_{X_1X_2Y_2}\, q_{X_2X_3Y_3} \dots\; q_{X_{N-1}X_NY_N}$$

⇨ *A product of symbol emission probabilities*

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joint probability of an observation sequence and a state sequence for the given model

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⇨ ***A product of symbol emission probabilities***

$$P(X\,|\lambda) = \pi\, p_{X_0X_1} p_{X_1X_2} p_{X_2X_3} p_{X_3X_4} \cdots\; P_{X_{N-1}X_N}$$

⇨ ***A markov chain evolution probability***

Exercise 2 revisited

Consider the same Markov Chain from exercise 1:

- State alphabet: $\boldsymbol{S} = \{s_1, s_2, s_3\}$
- Initial probability distribution: $\pi = [1, 0, 0]$
- Transition matrix: $\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0.0 \\ 0.0 & 0.6 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$

Exercise 2 revisited

But now, consider the binary observation alphabet: $\boldsymbol{O} = \{A\,,\, B\}$

And the following symbol emission probabilities:

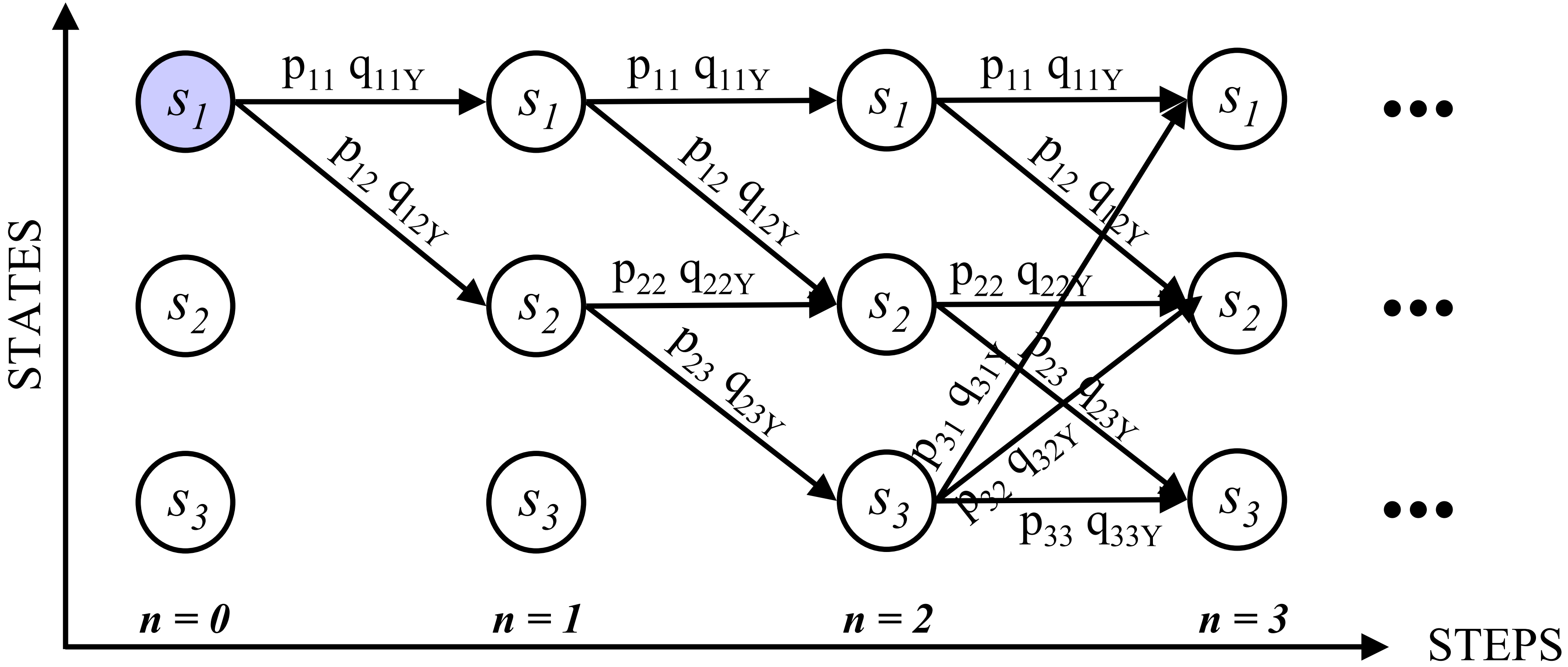
$$q_{ijA} = q(\,A \mid X_n,\, X_{n\text{-}1}) \qquad q_{ijB} = q(\,B \mid X_n,\, X_{n\text{-}1})$$

$$\begin{pmatrix} 1.0 & 0.5 & 0.5 \\ 0.5 & 1.0 & 0.5 \\ 0.5 & 0.5 & 1.0 \end{pmatrix} \qquad \begin{pmatrix} 0.0 & 0.5 & 0.5 \\ 0.5 & 0.0 & 0.5 \\ 0.5 & 0.5 & 0.0 \end{pmatrix}$$

What is the probability of observing the sequence $\boldsymbol{Y} = A,\, A,\, A$?

Lattice representation for HMMS

Consider the following representation for the HMM in exercise 2



Computation of probabilities

The probability of any observation sequence ***Y*** is computed by:

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Computation of probabilities

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- considering the corresponding symbol emission probabilities at each step in the sequence,
- computing the probabilities for all possible paths starting at $n=0$ and ending at $n=N$ (being N the size of sequence $\boldsymbol{Y}$),

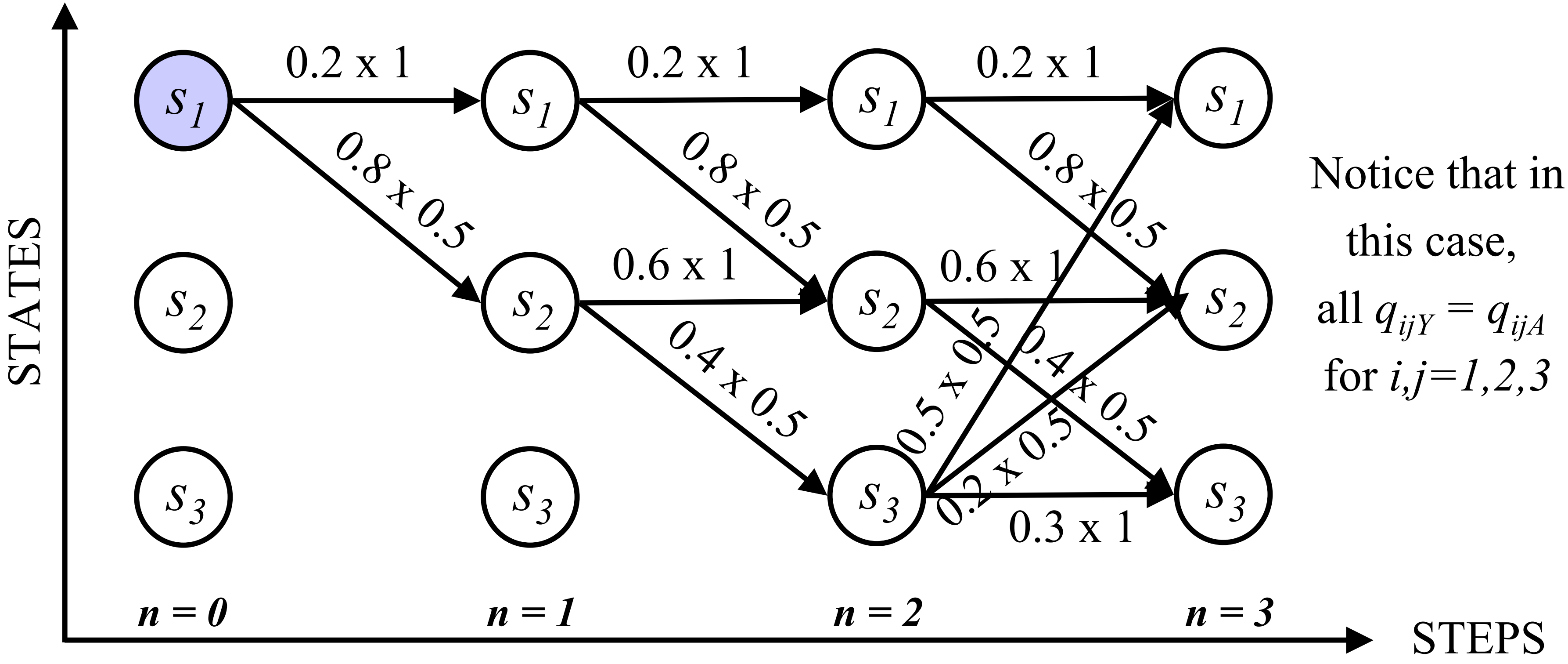
Computation of probabilities

The probability of any observation sequence Y is computed by:

- considering the corresponding symbol emission probabilities at each step in the sequence,
- computing the probabilities for all possible paths starting at $n=0$ and ending at $n=N$ (being N the size of sequence Y),
- adding up all resulting probabilities.

Computation of probabilities

So, in the particular case of $\boldsymbol{Y} = A, A, A$



Notice that in this case, all $q_{ijY} = q_{ijA}$ for $i,j=1,2,3$

Computation of probabilities

A total of nine different paths are possible:

$S_1\ S_1\ S_1\ S_1$: $p(path_1) = 0.2 \text{ x } 1.0 \text{ x } 0.2 \text{ x } 1.0 \text{ x } 0.2 \text{ x } 1.0 = 0.008$

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$S_1\ S_1\ S_1\ S_2$: $p(path_2) = 0.2 \text{ x } 1.0 \text{ x } 0.2 \text{ x } 1.0 \text{ x } 0.8 \text{ x } 0.5 = 0.016$

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Computation of probabilities

A total of nine different paths are possible:

$S_1\ S_1\ S_1\ S_1$: $p(path_1) = 0.2 \times 1.0 \times 0.2 \times 1.0 \times 0.2 \times 1.0 = 0.008$

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Computation of probabilities

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$S_1\ S_2\ S_3\ S_3$: $p(path_9) = 0.8 \text{ x } 0.5 \text{ x } 0.4 \text{ x } 0.5 \text{ x } 0.3 \text{ x } 1.0 = 0.024$

Computation of probabilities

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$$\left.\begin{array}{l}
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S_1\ S_2\ S_3\ S_2:\ p(path_8) = 0.8 \text{ x } 0.5 \text{ x } 0.4 \text{ x } 0.5 \text{ x } 0.2 \text{ x } 0.5 = 0.008 \\
S_1\ S_2\ S_3\ S_3:\ p(path_9) = 0.8 \text{ x } 0.5 \text{ x } 0.4 \text{ x } 0.5 \text{ x } 0.3 \text{ x } 1.0 = 0.024
\end{array}\right\} \begin{array}{l} P(A,A,A) \\ = 0.332 \end{array}$$

Computational complexity

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Computational complexity

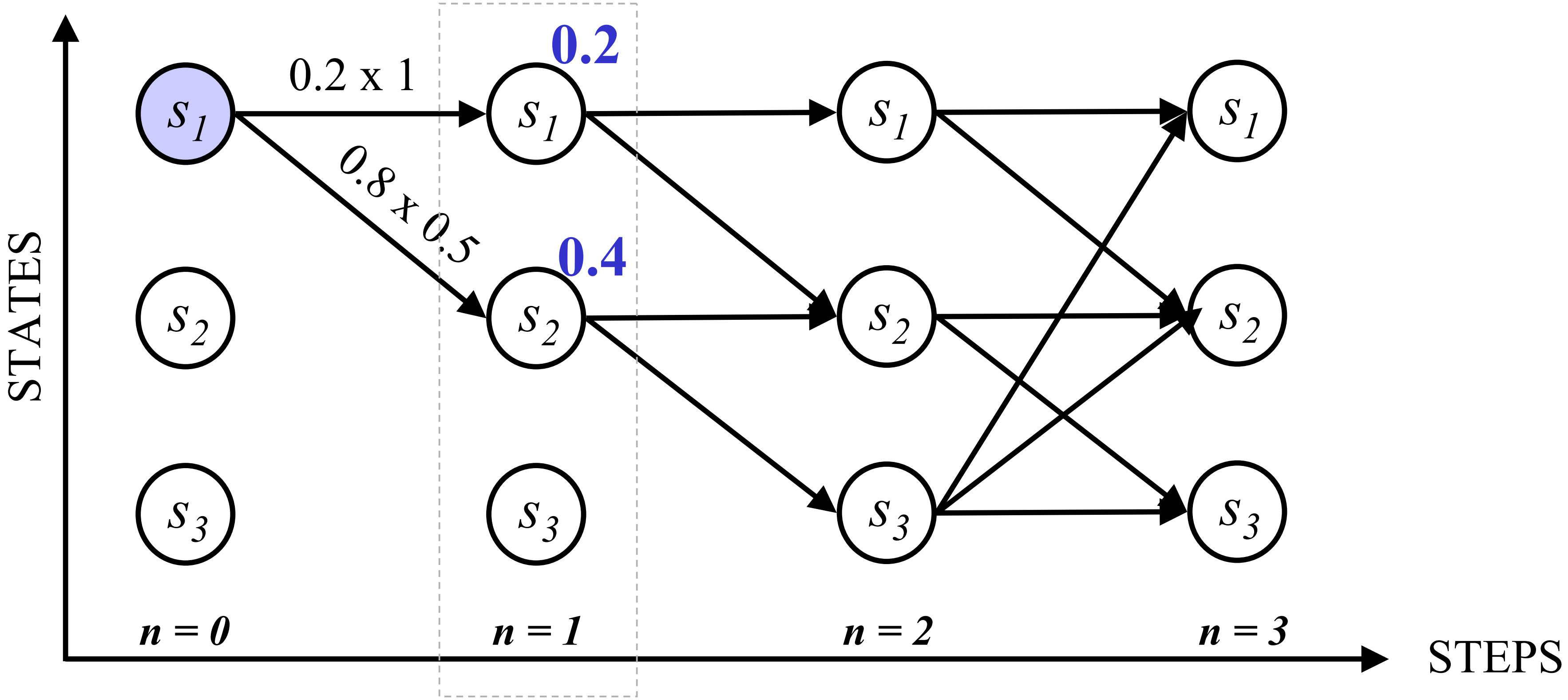
In the general case of an observation sequence of size N and a HMM of M states, the total number of multiplications to be performed is: $(2N+1)\, M^{N+1}$

- This problem becomes intractable for large values of N and M.
- However, it can be noticed that in the process there are some products that are repeated again and again...
- ***Dynamic programming*** consists in memorizing some partial results in order to avoid repeating the same computations !!!

Dynamic Programming

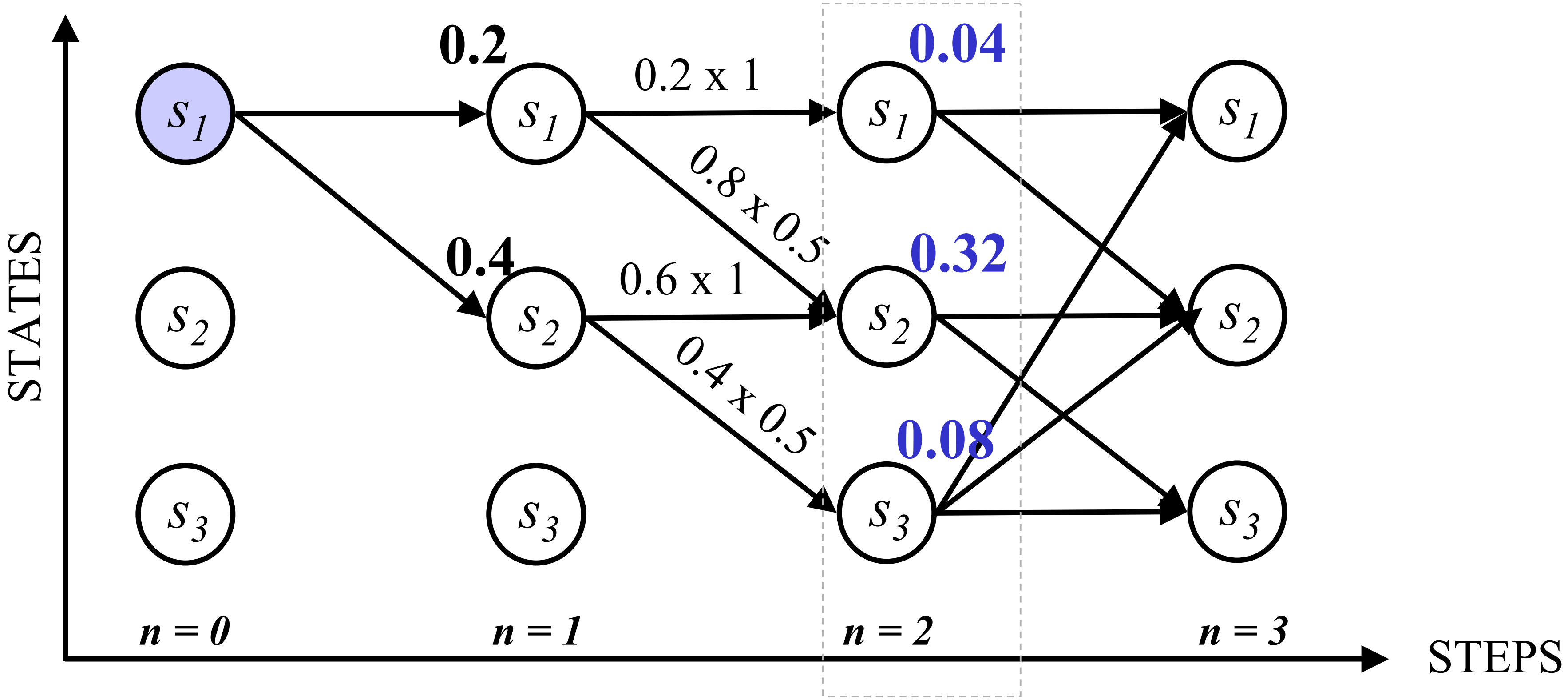
Consider exercise 2 again:

First we compute partial probabilities at $n=1$



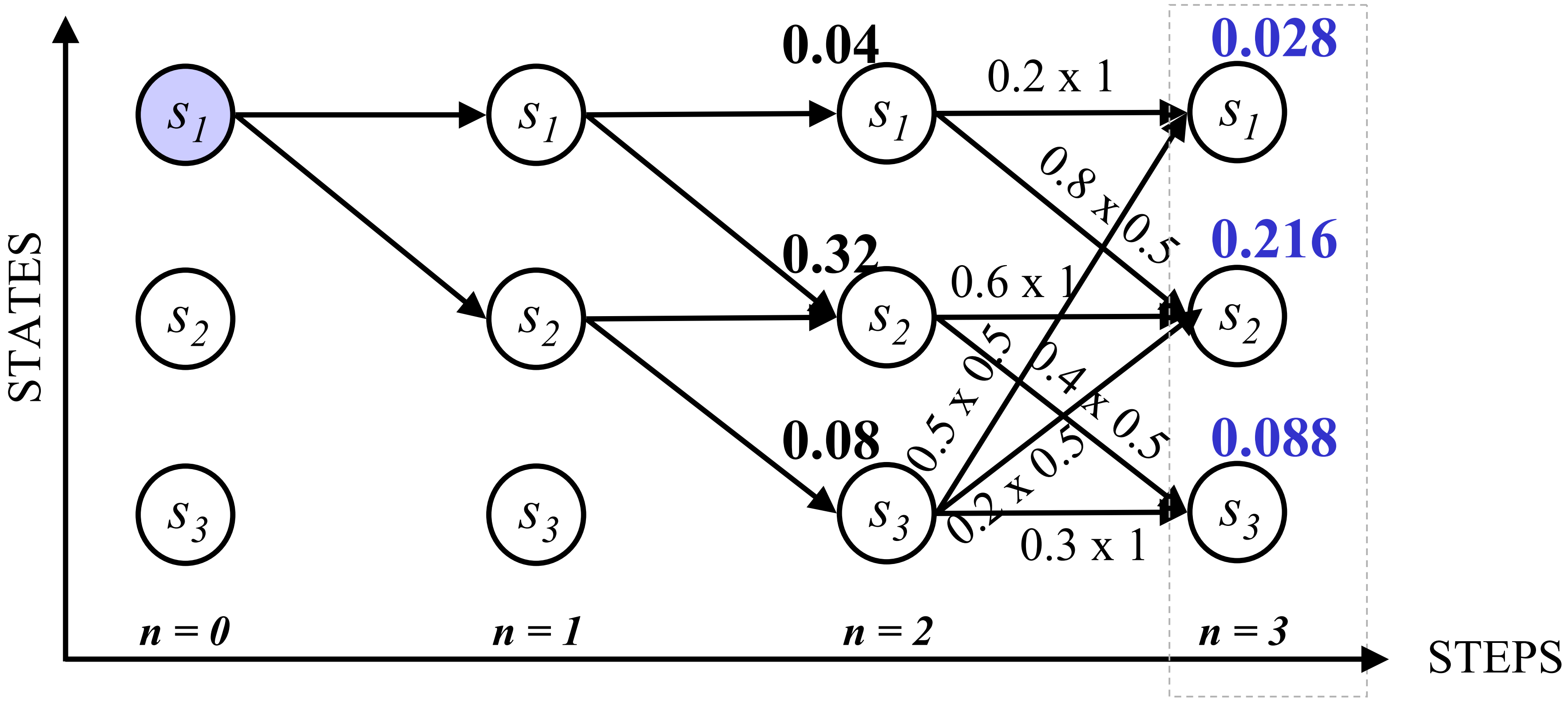
Dynamic Programming

Then, we proceed to compute partial probabilities at *n=2*



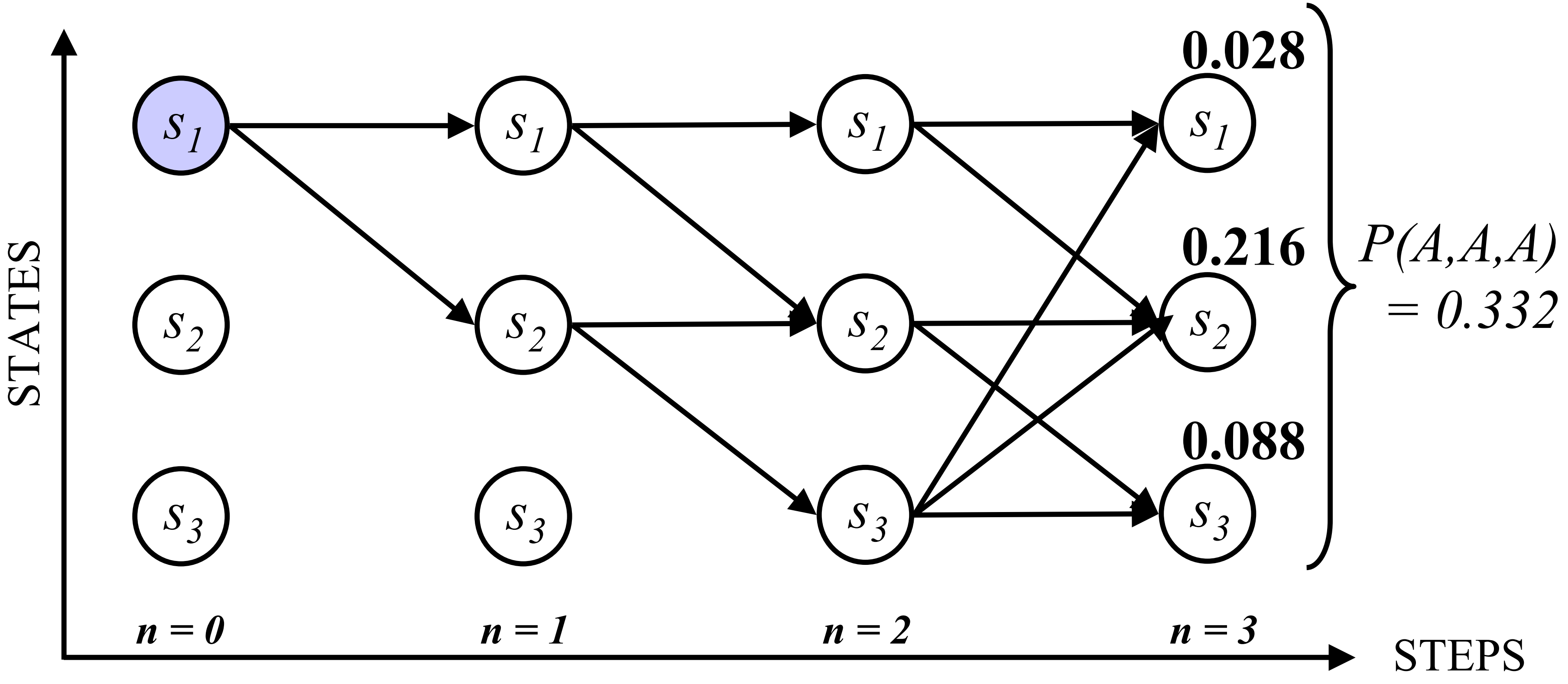
Dynamic Programming

And so on...



Dynamic Programming

Finally, the observation probability is computed by adding up probabilities at *n=3*



Dynamic Programming procedures

- The shown algorithm is referred to as the forward procedure for dynamic programming since it computes partial probabilities from the initial step at $n=0$ up to the final one at $n=N$

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Dynamic Programming procedures

- The shown algorithm is referred to as the forward procedure for dynamic programming since it computes partial probabilities from the initial step at $n=0$ up to the final one at $n=N$
- It is also possible to implement a backward procedure, in which partial probabilities are computed from the final step $n=N$ down to the initial one $n=0$.
- Dynamic programming procedures reduce the total number of multiplications required from $(2N+1)\ M^{N+1}$ to $2N\ M^{2}$

The forward procedure

- Consider the vector f_n which contains the probability values of having observed the sequence $Y_1, Y_2 \ldots Y_n$ and reaching any of the M states at step n. (f_n is a row vector of M elements)

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- Initialization: $f_0 = \pi$
- Induction: $f_n = f_{n-1} * \; (\mathbf{P} \, .* \, \mathbf{Q}_{\boldsymbol{Y_n}})$ for $n = 1, 2, 3 \ldots. N$
- Total probability: $P(Y|\,\lambda) = f_N * \boldsymbol{1}$

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- Initialization: $f_0 = \pi$

matrix product

MATLAB's dot product (element by element)

- Induction: $f_n = f_{n-1} * \ (\mathbf{P} \,.\!* \mathbf{Q}_{Y_n})$ for $n = 1, 2, 3 \ldots. N$

Symbol emission probability matrix for given observation at step *n*

- Total probability: $P(Y|\ \lambda) = f_N * \mathbf{1}$

column vector of ones

The backward procedure

- Consider the vector b_n which contains the probability values of observing the future sequence $Y_{n+1}, Y_{n+2} \ldots Y_N$ when being at any of the M states at step n. (b_n is a column vector of M elements)

The backward procedure

- Consider the vector b_n which contains the probability values of observing the future sequence Y_{n+1}, $Y_{n+2} \dots Y_N$ when being at any of the M states at step n. (b_n is a column vector of M elements)

- Initialization: $b_N = 1$

- Induction: $b_{n-1} = (\mathbf{P} \,.\!* \mathbf{Q}_{\mathbf{Y}_n}) * b_n$ for $n = N, N\text{-}1, N\text{-}2 \dots 1$

- Total probability: $P(Y|\,\lambda) = \pi * b_0$

The backward procedure

- Consider the vector b_n which contains the probability values of observing the future sequence $Y_{n+1}, Y_{n+2} \dots Y_N$ when being at any of the M states at step n. (b_n is a column vector of M elements)

column vector of ones

- Initialization: $b_N = 1$

Symbol emission probability matrix for given observation at step *n*

- Induction: $b_{n-1} = (\mathbf{P} \; .* \; \mathbf{Q}_{\mathbf{Y}_n}) * b_n$ for $n = N, N\text{-}1, N\text{-}2 \dots. 1$

- Total probability: $P(Y|\,\lambda) = \pi * b_0$

MATLAB's dot product

Important property

Consider the row vectors f_n and the column vectors b_n of the forward and backward procedures of dynamic programming:

- it can be proved that $P(Y|\lambda) = f_n * b_n$
- and it holds for all values of $n = 0,1,2 \dots N$

Exercise 3

Consider again the HMM of exercise 2...

1.- Use MATLAB or any other language to implement the forward and backward procedures for computing the probability of the following observation sequences:

a.- $\boldsymbol{Y} = A, A, A$ b.- $\boldsymbol{Y} = B, B, B$ c.- $\boldsymbol{Y} = B, A, B, A$

2.- Verify that $P(Y|\, \lambda) = f_n * b_n$ holds for any value of $n=0,1,2,\ldots N$

3.- Repeat steps 1.- and 2.- by taking $\pi = [0.2, 0.5, 0.3]$ this time

Hidden Markov Models

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Session 2: Dynamic Programming & HMMs

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