Sampling Theory Fundamentals

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I. Introduction

In this chapter we present some of the fundamental concepts of sampling theory that will be used later for explaining and understanding some of the basic issues related to the problem of seismic acquisition.

Sampling theory provides us with the basic elements for the successful design and implementation of a seismic survey. Indeed, seismic surveying, as will be seen later, has to be with the sampling of a seismic wave field that propagates through an earthen formation.

First, we will discuss some basic concepts on signals and their representations. Then, the sampling process and its effects will be studied from a mathematical point of view in order to better understand the phenomenon of aliasing and the problem of information preservation; as well as the problem of reconstruction. Finally, some related topics are presented and briefly discussed.

II. Types of Signals and their Representation

Most of the information we deal with everyday is related in one way or another to signals. More than being the information itself, signals are the means for transmitting or representing it. Every signal can be mathematically represented by a relation between two sets; a domain set, which is referred to as the variable, and a range set, which is referred to as the amplitude.

A. Types of Signals

According to the class of sets over which a signal can be represented, four different types of signals can be identified. They are illustrated in Figure 1 and explained next:

- An analog signal is a signal which is both continuous-variable and continuous-amplitude. In general, most of the signals encountered in nature belong to this class. This type of signals can be well represented by a continuous variable function $f(x)$ with x Real.
- A discrete-variable signal is a signal which is discrete-variable but continuous-amplitude. They are mathematically handled as numerical sequences and represented as $f[n]$ with n Integer[1].
- A discrete-amplitude signal is a signal which is continuous-variable but discrete-amplitude.
- A digital signal is a signal that is both discrete-variable and discrete-amplitude. Digital signals are perfectly suited for being codified in order to be handled by digital computers.

[1]A common mistake is to see a sequence $f[n]$ as a function $f(x)$ which is zero for every non-integer value of x. It is important, then, to make it clear that a sequence is only defined for integer values of its argument.

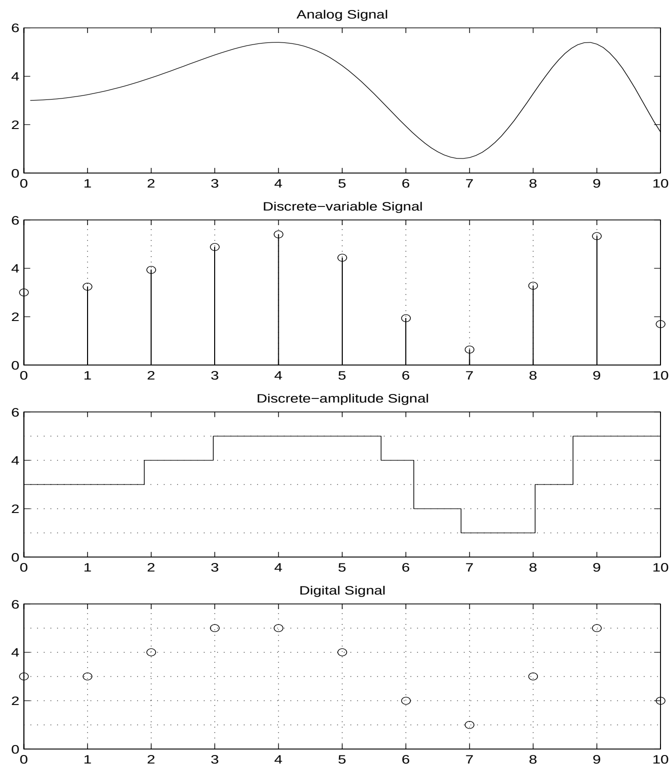

Fig. 1. Types of signals according to their domain and range sets.

Signals can also be classified according to the amount of variable dimensions and amplitude dimensions they are defined over. In this way a signal can be multivariable, if its domain is defined over more than one variable; and vectorial, if its range involves more than one parameter.

They also can be classified according to the type of variable or amplitude magnitudes under consideration. In this way there are signals in time or space or signals of voltage or current among others.

For example a TV program, as we see it, is an analog multivariable vector signal whose variables are space[2] and time and whose associated magnitudes are audio, brightness and color. However, the transmitted TV signal, which is referred to as the composed video signal, is a continuous-time signal of electric current, which is broadcasted by means of another multivariable vector signal, an electromagnetic wave field.

B. Equivalent Representation of Signals

In the same way sets and functions admit equivalent mathematical representations, a given signal can also be

[2]Although strictly speaking, a TV image is discrete in the vertical spatial dimension.

represented by means of different types of signals. For example, an audio signal can be represented by an equivalent electric signal and vice versa – this is the base of telephony.

Of special interest nowadays is the representation of analog signals by means of digital ones. This is because, although digital computers are powerful tools for handling and processing information, most of the signals we are generally interested in deal with are analog.

Although apparently trivial, the problem of representing analog signals with digital ones is not a simple one. This is basically because of two reasons: first, it is a non-linear process; and second, it is an irreversible process. However, as seen latter, some restrictions can be applied and some considerations taken into account in order to eliminate the ambiguities inherent to the digitization process.

To obtain a digital representation of an analog signal, two steps must be performed. First, the analog signal must be discretized in its variable dimension. This step is called sampling. Second, the resulting discrete-variable signal must be discretized in its amplitude dimension. This step is called quantization. Figure 2 illustrates the digitization process of an audio signal. As seen from the figure, an additional codification step is implemented in order to put the quantized samples into the binary format used by computers.

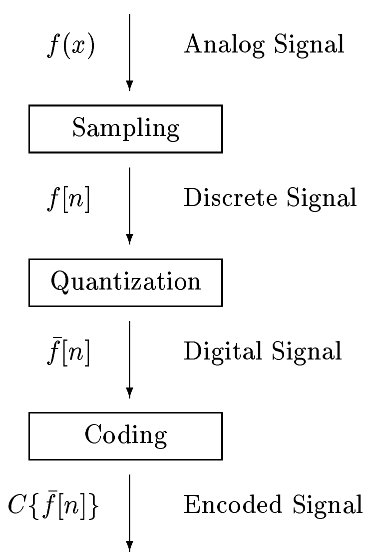

Fig. 2. Steps of the Digitization Process.

Although during both sampling and quantization some amount of information of the original analog signal is always lost, the sampling process is in general most critical than the quantization one. This is because quantization effects are generally neglectable thanks to the great numerical precision of modern computers[3]. For this reason, we will focus our attention on the study of sampling and discrete-variable signals only. For more information on quantization and its effects the reader can refer to [5].

[3]However, quantization effects must be taken into account and dealt with when recursive procedures, susceptible to error accumulation, are involved.

III. The Nyquist-Shannon Sampling Theorem

One of the most revolutionary concepts in communication theory has been the Nyquist-Shannon sampling theorem, [1] and [2]. According to this theorem, a band limited signal has a restricted number of degrees of freedom per variable interval. In other words, what the theorem states is that a band limited analog signal has an inherent redundancy of the information it contains. In this way, it is possible to represent the analog signal by using samples of it without lossing information. However, there is a limit on the sampling period to be used which is related to the band-width of the signal.

A. Theorem Postulate

The Nyquist-Shannon sampling theorem establishes the limits for preserving information when sampling band limited analog signals. It can be stated as follows: *A band limited signal with maximum spectral content W, can be sampled at a minimum rate of 2W without lossing any information.*

B. Mathematical Motivation

Consider a periodic function $f(x)$ with period $2W$. As known, it admits as Fourier series representation given by

$$f(x) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j\frac{2\pi n x}{2W}}\ , \tag{1}$$

where the coefficients $g[n]$ can be obtained from the Fourier analysis formula,

$$g[n] = \frac{1}{2W}\int_{-W}^{W} f(x)\, e^{j\frac{2\pi n x}{2W}}\, dx\,. \tag{2}$$

On the other hand, let us consider a continuous function $r(y)$ with spectral content $R(x)$. Through their Fourier transform relationship, we have that

$$r(y) = \int_{-\infty}^{\infty} R(x)\, e^{j2\pi y x}\, dx\,. \tag{3}$$

Now, without loosing generality, suppose that $R(x)$ is given by

$$R(x) = \begin{cases} f(x) & \text{if } -W \leq x \leq W \\ 0 & \text{Otherwise} \end{cases} \tag{4}$$

where $f(x)$ is the function previously defined in (1).

Now, suppose that the function $r(y)$ is uniformly sampled at the values of $y = n/2W$ for $n = \ldots -2, -1, 0, 1, 2 \ldots$ After doing this, and taking into consideration equation (4), the following discrete-variable representation of $r(y)$ can be obtained from (3);

$$r[n] = r\left(\frac{n}{2W}\right) = \int_{-W}^{W} f(x)\, e^{j\frac{2\pi n x}{2W}}\, dx\,. \tag{5}$$

Notice that the integral in (5) is the same, except for a scaling factor, as the one in (2). Indeed, it follows that $2W\, g[n] = r[n]$.

At this point we can see that, since there is a unique relationship between the function $r(y)$ and its Fourier transform $R(x)$, and between the function $f(x)$ and its Fourier series coefficients $g[n]$; we will be able to recover $r(y)$ from the values of $g[n]$ if, and only if, we are able to obtain $R(x)$ from $f(x)$. This ideas are illustrated in Figure 3

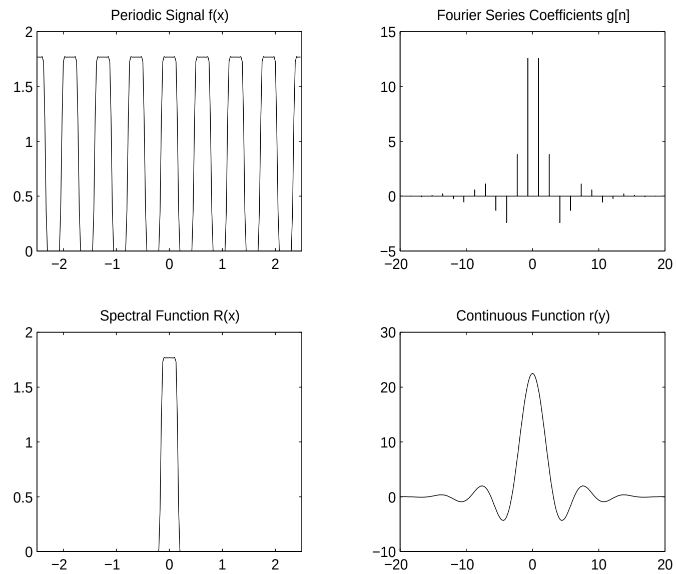

Fig. 3. Illustration of the Nyquist-Shannon Theorem.

IV. The Sampling Process

As described in the previous section, the Nyquist-Shannon theorem provides a basis for sampling properly analog signals in order to preserve information. This section presents a mathematical representation[4] of the sampling process and discusses it in detail.

A. Analog to Discrete Conversion

The mathematical representation of the sampling process is achieved by taking advantage of the Dirac's delta function. Suppose that the signal to be sampled, $f(x)$, is multiplied by a periodic unit impulse train, the resulting signal will be another periodic impulse train, $f_s(x)$, with the amplitude values of the original signal at the impulse locations;

$$f_s(x) = f(x) \sum_{n=-\infty}^{\infty} \delta(x - n x_s)\,, \tag{6}$$

where x_s is referred to as the sampling period.

It is important to notice that the resulting signal, $f_s(x)$, is still an analog signal. The desired discrete-variable signal, the sequence $f[n]$, is obtained from $f_s(x)$ by a subsequent conversion process that transforms the continuous impulse train into a discrete-variable sequence.

[4]Due to practical reasons, the physical implementation of sampling is actually different to the mathematical approach. For more information on sampler circuits, the reader can refer to [6]

By using the sifting property of the Dirac's delta function, equation (6) can be equivalently written as

$$f_s(x) = \sum_{n=-\infty}^{\infty} f(n x_s) \delta(x - n x_s)\,, \tag{7}$$

from which the conversion to a discrete sequence follows immediately,

$$f[n] = f(n x_s)\,. \tag{8}$$

Notice that this conversion process implicitly includes the normalization of the variable dimension by the sampling period value x_s. As will be seen in the next subsection, this normalization is also carried over to the spectrum of the sequence $f[n]$.

The mathematical model for the analog to discrete conversion process is illustrated in Figure 4.

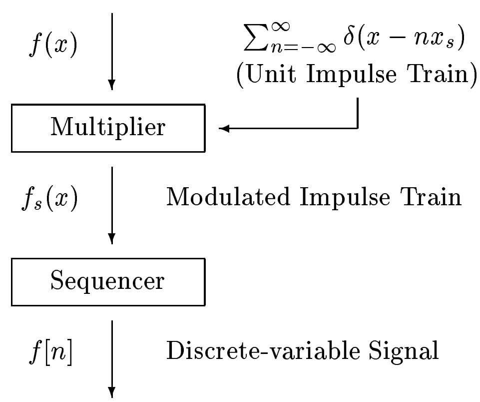

Fig. 4. Analog to Discrete Conversion Process.

B. Spectral Representation of Sampling

In order to have a better understanding of the effects of sampling, let us consider the problem in the frequency domain. By taking the Fourier transform of both sides of equation (6) and considering the convolution property of the transform, we get

$$F_s(\Omega) = \frac{1}{x_s}\, F(\Omega) * \sum_{k=-\infty}^{\infty} \delta(x - k\, 2\pi / x_s)\,, \tag{9}$$

which expresses the spectrum of the signal $f_s(x)$ in terms of the convolution between the spectrum of the original signal, $f(x)$, and the spectrum of the periodic unit impulse train (which is another impulse train [4]).

The value $2\pi/x_s$ in equation (9) is referred to as the sampling angular frequency and it is denoted by Ω_s. Notice also the presence of the scaling factor $1/x_s$.

By applying the convolution property of the Dirac's delta function, equation (9) can be expressed as

$$F_s(\Omega) = \frac{1}{x_s} \sum_{k=-\infty}^{\infty} F(\Omega - k\, \Omega_s)\,. \tag{10}$$

Equation (10) reveals that the spectrum of the modulated impulse train $f_s(x)$ is actually given by the summation of scaled copies of the original signal's spectrum located at integer multiples of the sampling angular frequency Ω_s. The spectrum of the related discrete sequence, $f[n]$, can be obtained by using the same normalization that took place in the conversion step presented in (8). In this way, the angular frequency axis, Ω, is normalized as

$$\omega = \Omega\, x_s = \frac{2\pi}{x}\, x_s \,. \tag{11}$$

Figure 5 shows a continuous signal $f(x)$, a related impulse train $f_s(x)$ and its corresponding discrete sequence $f[n]$; together with their corresponding amplitude spectra.

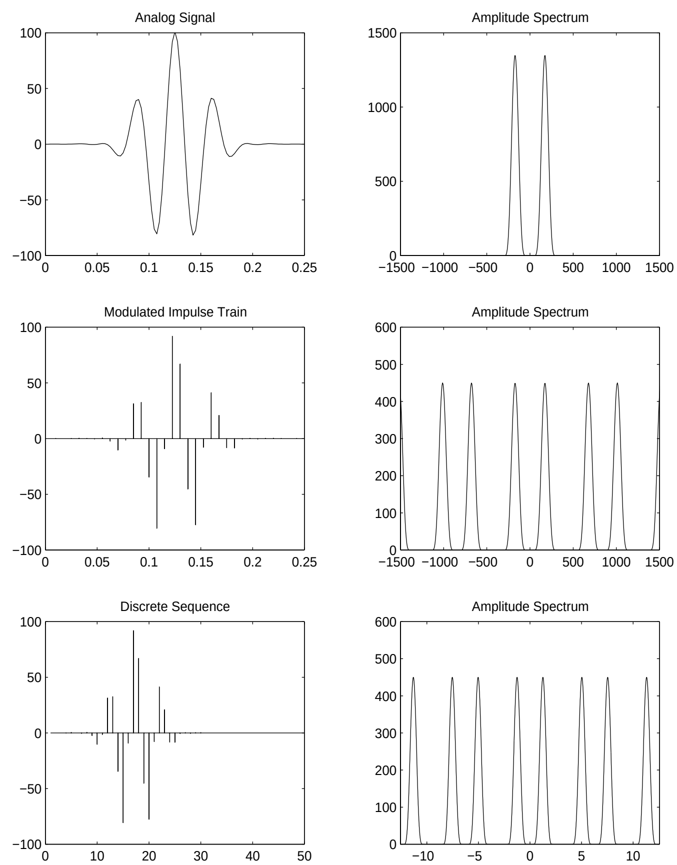

Fig. 5. Continuos, Impulsive and Discrete Signals and Spectra.

Notice from (11) that the value of the sampling angular frequency, Ω_s, corresponds to $\omega = 2\pi$ in the new normalized frequency axis ω. As a consequence, the spectrum, or frequency response, of the discrete sequence $f[n]$ is a periodic function with period 2π. This fact is emphasized in the standard notation by denoting the Fourier transform of such a discrete sequence as $F(e^{j\omega})$, which is given by

$$F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f[n]\, e^{-j\omega n} \,. \tag{12}$$

It can be noticed that (12) is totally equivalent to equation (1) when $W = \pi$. Without loosing generality, the related synthesis formula can be obtained from equation (2). It is given by

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\omega})\, e^{j\omega n}\, d\omega \,. \tag{13}$$

Equations (12) and (13) constitute the analysis and synthesis Fourier formulas for discrete sequences [4].

V. The Reconstruction Problem

This section studies in detail the problem of reconstructing an analog signal from a discrete-variable representation of it. First, the discrete to analog conversion process is presented. Afterwards, the phenomenon of aliasing and its implications in the problem of preserving information is discussed.

A. Discrete to Analog Conversion

Regardless of the origin of a discrete-variable signal $f[n]$, an analog representation of it, $\hat{f}(x)$, can be always obtained. The problem of finding such an analog representation is indeed an interpolation problem, in which a continuous-variable signal is generated from the discrete-variable one. This interpolation can be performed by following the two-step process illustrated in Figure 6, which is described next.

First, the discrete sequence must be converted into a periodic impulse train of the form

$$\hat{f}_s(x) = \sum_{n=-\infty}^{\infty} f[n]\, \delta(x - n x_p) \,, \tag{14}$$

where x_p defines the periodicity of the resulting impulse train.

Second, the obtained signal, $\hat{f}_s(x)$, is filtered in order to obtain the desired analog representation $\hat{f}(x)$. Although any filtering method can be used in principle, linear filtering is the most common technique used in practice. In this way, the desired representation will have the form of a convolution,

$$\hat{f}(x) = \hat{f}_s(x) \,*\, g(x) \,, \tag{15}$$

where $g(x)$ represents the impulse response of the reconstruction filter.

There exists, in the signal processing theory, an optimal reconstruction filter for performing the convolution in (15). It is commonly referred to as the ideal reconstruction filter, $g_{ideal}(x)$, and its impulse response is given by

$$g_{ideal}(x) = \frac{sin(\pi x / x_p)}{\pi x / x_p} \,. \tag{16}$$

The reason of this will become evident in the next subsection, where a frequency domain representation is considered again.

Of particular interest is the case of recovering an analog signal $f(x)$ from a discrete representation of it $f[n]$. This reconstruction problem is solvable by means of the methodology described above only if the following two conditions

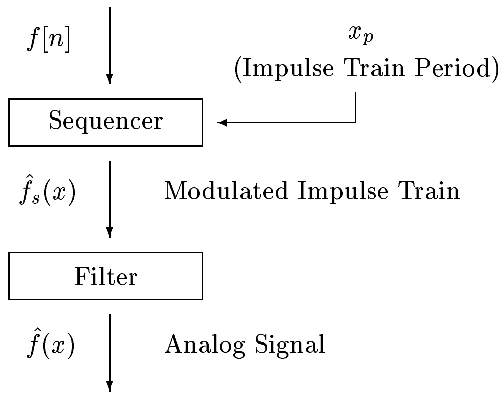

Fig. 6. Discrete to Analog Conversion Process.

are met. First, the sampling period x_s used for obtaining $f[n]$ must be known, and $x_p = x_s$ must be used in the reconstruction process. Second, the information preservation conditions established in the Nyquist-Shannon theorem must be met.

B. Information Preservation and Aliasing

As was seen in Figure 5, the spectral content of the sequence $f[n]$ corresponds to a periodic version of the spectrum of the original analog signal $f(x)$. It is evident, from the figure, that the original analog signal's spectrum can be recovered from the periodic one by first, denormalizing the frequency axis according to the sampling period x_s; and second, retaining the appropriate period of it. This last step can be easily achieved by using a ideal low pass filter with spectral content

$$G_{ideal}(\Omega) = \begin{cases} x_s, & |\Omega| < \pi/x_s \\ 0, & \text{Otherwise} \end{cases} \tag{17}$$

which happens to be the spectral content of the ideal reconstruction filter's impulse response, $g_{ideal}(x)$, presented in equation (16).

At this point, it is important to mention that due to practical reasons, the implementation of an ideal filter such as the one presented in (17) is not possible. However, it is always possible to design good approximations of it [?].

Figure 7 illustrates the ideas presented above for the same example previously presented in Figure 5. In the example, a sampling period of $x_s = 7.5\,10^{-3}$ was used. As can be seen from the figure, the maximum angular frequency content of the original signal is about $\Omega_{max} = 280$. Then, according to the Nyquist-Shannon theorem, the information in $f(x)$ is preserved during the sampling process since $x_s < 2\pi/(2\,\Omega_{max}) \approx 1.1\,10^{-2}$. Consequently, $f(x)$ can be recovered from $f[n]$.

Now, suppose that we increase the sampling period x_s towards the maximum possible value established by the Nyquist-Shannon theorem. According to equation (6) the period of the resulting sampled signal's spectrum will decrease and the spectral replicas of $F(\Omega)$ will get closer to each other. When the value of x_s exceeds the sampling theorem limiting condition, the spectral replicas get so close

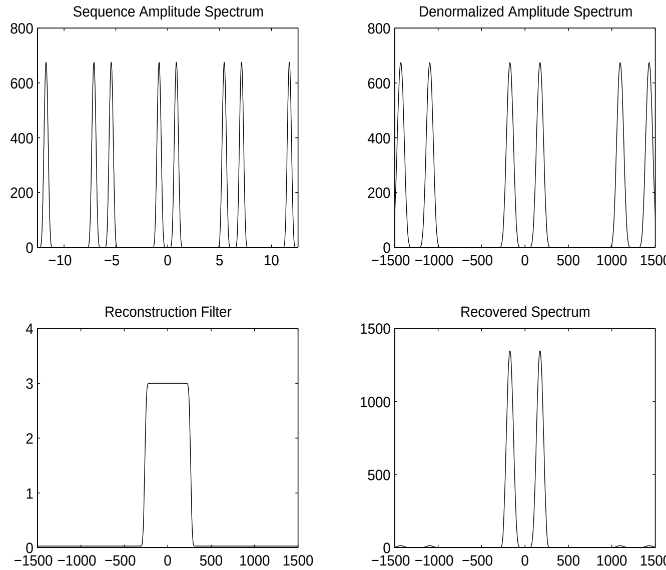

Fig. 7. Recovery of an Original Analog Signal's Spectrum.

that overlap to each other making it impossible to recover the original signal's spectral content. This phenomenon is referred to as aliasing, and when it occurs the information in the original signal becomes unrecoverable[5].

Figure 8 shows the amplitude spectra of various discrete sequences, obtained from the same analog signal in Figure 5, by using different sampling period values. Notice how spectral overlapping occurs when the sampling period x_s is greater than its critical value $1.1\,10^{-2}$.

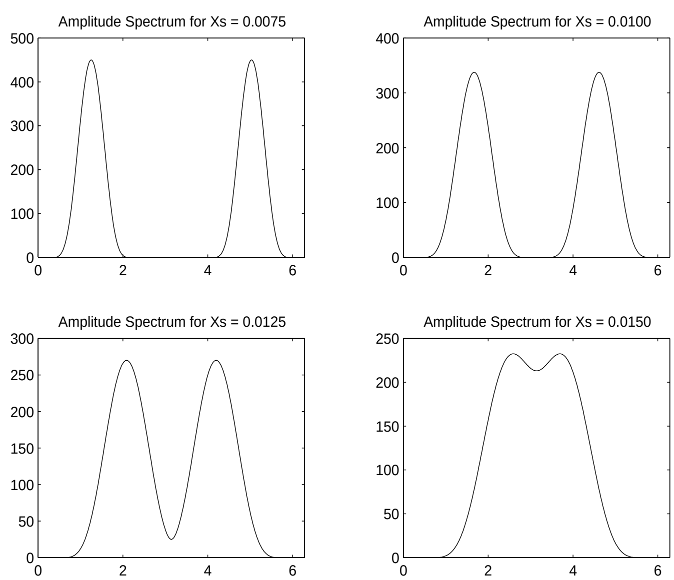

Fig. 8. Spectra for Sequences with Different Sampling Periods.

[5]However, under very special circumstances, aliased information can be recovered by a process of aliasing cancellation. Esteban and Galand first introduced this idea in 1978 with their Quadrature-Mirror-Filters [7]

As final remarks, it is important to mention that, in practice, it is always recommended to use sampling periods smaller than the critical one established in the Nyquist-Shannon theorem. This is because, as mentioned before, the implementation of the ideal reconstruction filter is not possible in practice. Then, in order to make it possible the recovery of the information by means of a non-ideal filter, it is necessary to leave some redundancy in the data by using a less-than-critical sampling period. However, there will always be a compromise in choosing the sampling period because the smaller it is, the larger the computational and storing requirements are.

Another important issue is the one of handling high frequency noise. In most cases, the noise has a broader spectral content than the signal of interest. As a result, when the sampling period is selected according to the signal's frequency content, the high frequency portion of the noise spectrum get in alias with the signal's spectrum after the sampling process. This problem is handled in practice by using an anti-aliasing filter, which consists in a low pass filter that is applied to the signal before sampling it.

VI. Related Topics

The most fundamentals concepts of sampling theory have been discussed and developed in the previous sections. This section presents a brief review of some additional topics that are related to the problem of sampling and can be of special interest in some specific problems that will be discussed in subsequent chapters.

First, the extrapolation of the sampling theory to multidimensional signals is discussed. Then, the problem of changing the sampling period of an already sampled signal is presented. And finally, the discrete-spectrum approach to handling discrete sequences and its implications are discussed.

A. Multidimensional Sampling

The problem of properly sampling multidimensional signals is based on the same concepts underlying the Nyquist-Shannon sampling theorem. If a unique sampling period is desired, it must be small enough such that spectral overlapping does not occur. However, since multidimensional signals often are defined over different parameter dimensions, it is very common to define a different sampling period for each of the dimensions. Nevertheless, the sampling theory conditions must be met for each of the involved dimensions in order to preserve the signal's information.

Just as in the case of one-dimensional signals, the selection of the sampling period obeys to a compromise between information preservation and cost. For example, in the case of movies, spatial sampling is much more critical than temporal sampling since when projecting the movie on a huge screen, grain effects are undesirable. On the other hand, temporal sampling is strongly restricted by the cost of film. For this reason, the smallest possible number of frames per second admitted by the human eye is used, which is 21 frames per second. Actually, the temporal sampling frequency used in movies is, by far, smaller than the critical one, producing then temporal aliasing[6]. Indeed, when watching a movie, the eye is being constantly tricked and the brain is performing an incredible interpolation task.

Another interesting example of multidimensional sampling is the case of seismic exploration geophysics. In this case, the seismic wave field is sampled in space and time at the earth surface. Opposite to the previous example, the strong economical restriction here is on spatial sampling. This is because the spatial sampling of the seismic wave field implies lying geophones all over the field under consideration, which can be, and it is indeed, very much expensive and time consuming. For this reason, it is commonly found in practice that seismic data lives in spatial aliasing, leaving in this case the reconstruction problem to the interpreter's imagination !

B. Changing the Sampling Period

In many practical applications, the problem of changing the sampling period of an already sampled signal is often found. Although this can be done by obtaining an analog representation of the sampled signal and resampling it with the desired new sampling period, this approach is not an efficient and desirable alternative. On the other hand, the problem of changing the sampling period can be totally handled in the discrete domain.

The process of changing the sampling period x_s to a larger one x_D is referred to as downsampling; while the opposite process, changing x_s to a smaller sampling period x_U, is referred to as upsampling. Figure 9 illustrates the two discrete systems used for performing downsampling and upsampling, a decimator and an interpolator respectively.

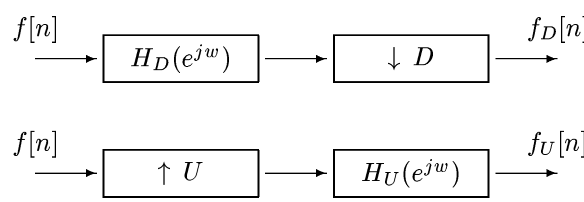

Fig. 9. Discrete Systems for Changing the Sampling Period.

As seen from the figure, a decimator is composed of a discrete anti-aliasing filter followed by a variable-domain compressor. The function of the compressor is to eliminate $D-1$ samples from each D samples. The parameter D is referred to as the compression factor, which determines the downsampling rate of the decimator. In this way, if the sampling period of the input signal $f[n]$ is x_s, the sampling period of the resulting signal $f_D[n]$ is $x_D = x_s\, D$.

On the other hand, an interpolator is composed of a variable-domain expander followed by a discrete reconstruction filter. The function of the expander is to add $U-1$ zero-valued samples between each two samples of

[6]This refers to the fact that the spectral content is in alias due to a deficient sampling in time. This must not be confused with the concept of aliasing in the time domain (variable domain), which will be discussed later.

$f[n]$. The parameter U is referred to as the expansion factor, which determines the upsampling rate of the interpolator. In this way, if the sampling period of the input signal $f[n]$ is x_s, the sampling period of the resulting signal $f_U[n]$ is $x_U = x_s/U$.

Changing the sampling period by a non-integer factor is also possible by a cascade combination of an interpolator followed by a decimator. In this case, the sampling period of the resulting signal would be $x_s\, D/U$; where x_s corresponds to the sampling period of the input signal $f[n]$.

C. Spectral Sampling

As was already seen from equation (12), the frequency response $F(e^{jw})$ of a discrete sequence $f[n]$ is a periodic continuous-variable function. In many computational applications of signal processing, operations are carried over on the frequency domain. Then, it is required to handle frequency responses in a discrete way.

This problem can be visualized as the problem of sampling the spectrum content of discrete sequences. Here too, the conditions of the Nyquist-Shannon theorem must be met in order to preserve information. By a similar analysis to the one performed when sampling signals in their variable domain, it can be seen that the risk of aliasing is also present but, in this case, it will occur as overlapping of replicas of the sequence in the variable domain. This phenomenon is referred to as aliasing in the variable domain.

When the sequence under consideration is of finite length[7] N, the use a sampling period of $2\pi/N$ is a sufficient condition for properly sampling its spectrum. In other words, given a sequence $f[n]$ of N samples, N samples of its frequency response $F(e^{jw})$ are enough to recover it. In fact that would be the critical sampling period established by the sampling theorem.

From a mathematical point of view, the result of this process is a couple of discrete periodic sequences of length N, $\tilde{f}[n]$ and $\tilde{F}[k]$, which are related to each other through the following two equations

$$\tilde{F}[k] = \sum_{n=0}^{N-1} \tilde{f}[n]\, e^{-j\,\frac{2\pi k}{N}\,n} \tag{18}$$

$$\tilde{f}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{F}[k]\, e^{j\,\frac{2\pi k}{N}\,n} \tag{19}$$

which are known as the discrete Fourier transform (DFT) analysis and synthesis formulas respectively. The tilde is often used in the standard notation in order to emphasize the periodic character of both sequences.

The two new sequences, $\tilde{f}[n]$ and $\tilde{F}[k]$, are related to the original sequence $f[n]$ and its frequency response $F(e^{jw})$ as follows: $\tilde{f}[n]$ is a periodic version of $f[n]$, and $\tilde{F}[k]$ is a sampled version of $F(e^{jw})$.

[7]Which is always the case in practical applications since it is not possible to handle infinite-length sequences in a digital computer.

VII. Summary

The Nyquist-Shannon sampling theorem presents the basis for properly sampling analog signals in order to obtain discrete representations without loosing any of the information contained in them. According to the theorem, a signal with a maximum frequency content of W must be sampled above the minimum sampling frequency of $2W$ in order to preserve information.

The analysis of the sampling process in the frequency domain give us an important insight of how the information is preserved or lost, being this latter the case in which spectral overlapping, or aliasing, occurs. When an analog signal has been properly sampled, it can be perfectly recovered from its related sequence by a process of interpolation.

It is important, in order to avoid confusion, to have it clear the differences among the different concepts that were presented. An analog signal $f(x)$, with Fourier transform $F(\Omega)$, is sampled to obtain a discrete representation of it, the sequence $f[n]$, with frequency response $F(e^{jw})$, which happens to be a periodic, scaled and frequency-normalized version of $F(\Omega)$. It is also possible to obtain an analog representation $\hat{f}(x)$ for a discrete sequence $f[n]$. If the sequence $f[n]$ was obtained from an original analog signal $f(x)$, the sampling period used is known and the Nyquist-Shannon theorem conditions were met, it is possible to obtain an analog representation $\hat{f}(x)$ such that $\hat{f}(x) = f(x)$.

All concepts considered here were based on the assumption that all samples are uniformly spaced. Multirate signal processing deals with discrete signals resulting from a non-uniform sampling process. For information on this topic, the reader must refer to [3].

VIII. Discussion

The following problems or exercises are strongly recommended in order to develop a better understanding of the concepts discussed before. Some of them might require further reading or research.

1.- Sampling Band-pass Signals

The postulate of the Nyquist-Shannon theorem for sampling analog signals actually imposes a sufficient condition but not a necessary one. In fact, when dealing with band-pass signals, it is often possible to find a sampling period that is larger than the one established by the theorem and still preserve the signal's information.

Suppose that we are dealing with a continuous-time band-pass signal with spectral content between $15\,Hz$ and $25\,Hz$. What are all the possible sampling period values such that aliasing does not occurs? How should the frequency response of the ideal reconstruction filter look like in this case?

Suppose the general case of a band-pass signal with spectral content between f_{min} and f_{max}. Under what conditions is it possible to sample it properly by using a sampling period larger than $1/f_{min}$? Can you postulate a generalized sampling theorem?

Consider again the band-pass signal with spectral content between $15\,Hz$ and $25\,Hz$. Additionally, suppose that it is a zero phase signal and its amplitude spectrum is symmetric around $20\,Hz$. In this particular case, what is the maximum possible sampling period?

2.- Decimator and Interpolator Filters

Consider the interpolator and decimator systems presented in Figure 9. According to the Nyquist-Shannon sampling theorem, how should their frequency responses, $H_D(e^{jw})$ and $H_U(e^{jw})$, be defined? Which are their corresponding impulse responses, $h_D[n]$ and $h_U[n]$?

Suppose that an interpolator is connected in cascade to a decimator in order to perform a non-integer sampling rate conversion and both filters are combined into a single one. Which are the frequency response, $H_{D/U}(e^{jw})$, and the impulse response, $h_{D/U}[n]$, of the new filter? What important property of the $Senc$ function can you infer from this result?

3.- Discrete Fourier Transform (DFT)

Verify that equations (18) and (19) constitute a transform pair. In order to do that, you will have to prove first the orthogonality property of complex exponentials, which is given by

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{-j\,\frac{2\pi k}{N}\,n} = \begin{cases} 1, & n/N\,\text{Integer} \\ 0, & \text{Otherwise.} \end{cases} \quad (20)$$

4.- The Ideal Filter Paradox

A very bright student claims that he has found a way for implementing ideal filtering by using discrete signal processing. The process is described as follows. First, the analog signal to be filtered must be sampled into a discrete sequence. Then, the discrete Fourier transform of the obtained sequence must be computed and multiplied by the discrete Fourier transform of the ideal filter frequency response. Notice that such a multiplication is equivalent to perform a convolution in the variable domain. Afterwards, the inverse discrete Fourier transform is applied to the obtained response; and finally, the obtained sequence is converted back to an analog signal.

We know from previous discussions that ideal filtering is not realizable in practice since it involves infinite length impulse responses. However, it seems that the procedure just described overcomes such a difficulty by using the properties of the DFT. Take some time to think about this paradox and decide if the smart student is actually right or wrong. In either of the cases justify your conclusion.

Hint: Consider the effects of zero-padding a sequence.

5.- Aliasing Cancellation

One of the most interesting problems in signal processing is the problem of aliasing cancellation. Let us illustrate it with a simple example.

Consider the sequence $g[n] = \{-1, 1, 3, 0, 3, 1, -1\}$, defined from $n = -3$ to $n = 3$, and the two digital filters $f_1 = \{1, 1\}$ and $f_2 = \{1, -1\}$. Compute the frequency responses $G(e^{jw})$, $F_1(e^{jw})$ and $F_2(e^{jw})$; and make a sketch of their respective amplitudes. Notice that $f_1[n]$ and $f_2[n]$ are low-pass and high-pass filters respectively with frequency responses very far from being ideal.

Now compute the discrete convolutions $g_1 = g * f_1$ and $g_2 = g * f_2$. Multiply the resulting sequences g_1 and g_2 by the comb sequence $c[n] = \{\ldots, 1, 0, 1, 0, 1, 0, 1, 0, \ldots\}$ to obtain the new sequences $\hat{g}_1$ and $\hat{g}_2$. Sketch the frequency responses of this two new sequences. Are their spectra affected by aliasing?

Compute two new discrete convolutions $\tilde{g}_1 = \hat{g}_1 * \hat{f}_1$ and $\tilde{g}_2 = \hat{g}_2 * \hat{f}_2$, where the new filters $\hat{f}_1$ and $\hat{f}_2$ are reverse-time versions of the original filters; i.e. $\hat{f}_1 = \{1, 1\}$ and $\hat{f}_2 = \{-1, 1\}$. Finally, add the resulting sequences in order to get $\tilde{g} = \tilde{g}_1 + \tilde{g}_2$. Can you find any relationship between the original sequence $g[n]$ and the obtained one $\tilde{g}[n]$? Can you find an explanation for that?

6.- Unavoidable Aliasing

Although for practical purposes aliasing can be considered to be avoidable by means of an appropriate sampling rate, from a mathematical point of view, aliasing only can be avoided if infinite length sequences are considered. In practice, due to computational restrictions, finite length sequences are always considered. Take some time to think about the implications of these ideas and try to justify the validity of practical discrete signal processing applications.

Consider now the DFT analysis, it seems that in this case aliasing is always going to be present either in the frequency domain or in the variable domain. What justification can you give?

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